Contributions to the continuity problem for Lyapunov exponents

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Abstract

The aim of this work is to study the continuity and semi-continuity of the Lyapunov exponents in two different contexts. The first one concerns linear cocycles on partially hyperbolic dynamics. It is known that the Lyapunov exponents can be very sensitive as functions of the cocycle. Example of this is the result of Bochi-Mañé which shows that every $SL(2,\mathbb{R})$-cocycle that is not uniformly hyperbolic can be approximated by another with zero exponents. We prove that the set of fiber-bunched $SL(2,\mathbb{R})$-valued Hölder cocycles with nonvanishing Lyapunov exponents over a volume preserving, accessible and center-bunched partially hyperbolic diffeomorphism is open. Moreover, we present an example showing that this is no longer true if we do not assume accessibility in the base dynamics. This is a joint work with Lucas Backes and Mauricio Poletti.

In the second part of this work we will restrict our attention to the study of llocally constant cocycles associated with probability distributions with non-compact support in $SL(2,\mathbb{R})$. Bocker-Viana proved that for distributions with compact support, the exponents vary continuously. We analyze the behavior of the Lyapunov exponents when the measures are not compact, showing that in this case, the Lyapunov exponents, considered as functions of the measure, are semi-continuous with respect to the Wasserstein topology but not the weak* topology. Moreover, we prove that they are not continuous relative to the Wasserstein topology.

Resumo

O objetivo deste trabalho é estudar a continuidade e a semi-continuidade dos expoentes de Lyapunov em dois contextos diferentes. O primeiro diz respeito a cociclos lineares sobre dinâmicas parcialmente hiperbólicas. É sabido que os expoentes de Lyapunov podem ser muito sensíveis como funções do cociclo. Exemplo disto é o resultado de Bochi-Mañé que mostra que todo $SL(2,\mathbb{R})$-cociclo contínuo que não é uniformemente hiperbólico pode ser aproximado por outro com expoentes nulos. Mostrarei que o conjunto dos $SL(2,\mathbb{R})$-cociclos "fiber-bunched"; com expoente de Lyapunov não nulos, sobre um difeomorfismo parcialmente hiperbólico, é um aberto. Este é um trabalho conjunto com Lucas Backes e Mauricio Poletti.

O segundo tipo de resultados trata de expoentes de Lyapunov de cociclos localmente constantes associados a distribuções de probabilidade com suporte não compacto em $SL(2,\mathbb{R})$. Bocker-Viana provaram que, para distribuições com suporte compacto, os expoentes variam continuamente. Analizarei o comportamente dos expoentes de Lyapunov quando as medidas têm suporte não compacto, mostrando que neste caso tem-se semi-continuidade com a topologia de Wasserstein, mas não na topologia fraca*. Além disso, não há continuidade mesmo na topologia de Wasserstein.

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Introduction

The theory of linear cocycles goes back to the works of Furstenberg, Kesten [15, 14] and Oseledets [22]. The simplest examples of linear cocycles are given by derivative transformations of smooth dynamical systems. The cocycle *generated* by $A(x) = Df(x)$ over f is called the *derivative cocycle*. Taking as an example the *hyperbolic theory* of Dynamical Systems where one can understand certain dynamical properties of f by studying the action of Df on the tangent space, one can hope that by studying properties of linear cocycles one can also deduce some properties of f. Nevertheless, the notion of linear cocycle is much more general and flexible, and arises naturally in many other situations as in the spectral theory of Schrödinger operators, for instance.

In the present work we are interested in the asymptotic behavior of $A^n(x)$. Thus, we are interested in understanding certain regularity properties of *Lyapunov exponents*. The Lyapunov exponents are quantities that measure the average exponential growth of the norm iterates of the cocycle along invariant subspaces on the fibers. They describe the chaotic behavior of the system. For example, a strictly positive maximal Lyapunov exponent is synonymous of exponential instability. It is an indication that the system modeled by the cocycle behaves chaotically, and the maximal Lyapunov exponent measures the chaos. These objects are one of the most fundamental notions in dynamical systems.

It is well known that, in general, Lyapunov exponents can be very sensitive as functions of the cocycle. For instance, Bochi [7, 8] proved that in the space of $SL(2, \mathbb{R})$-valued continuous cocycles over an aperiodic map, if a cocycle is not hyperbolic, then it can be approximated by cocycles with zero Lyapunov exponents. In particular, there are cocycles with positive Lyapunov exponents that are accumulated by cocycles with zero Lyapunov exponents.

Furthermore, when the base dynamic is far from being hyperbolic, for example, when f is a rotation on the circle, Wang and You [26], showed that having non-zero Lyapunov exponents is not an open property even in the C^∞ topology.

Bocker and Viana [9] constructed an example over a hyperbolic map showing that the same phenomenon can happen in the Hölder realm. In order to construct their example, Bocker and Viana exploited the fact that the cocycle is not *fiber-bunched*. In fact, it was shown by Backes, Butler and Brown [5] that in the fiber-bunched setting over a hyperbolic map the Lyapunov exponents vary continuously with respect to the cocycle and, in particular, cocycles with positive Lyapunov exponents can not be approximate by cocycles with zero Lyapunov exponents.

In the first part of this work we are interested in understanding the case when the cocycle still have some regularity properties, namely, it is fiber-bunched but the base dynamics exhibit some mixed behaviour of hyperbolicity and non-hyperbolicity, that is, the map f is *partially hyperbolic*. More precisely, f over a compact manifold M is such that there exists a nontrivial, Df-invariant splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$ and a Riemannian metric on M such that vectors in E^s are uniformly contracted by Df in this metric, vectors in E^u are uniformly expanded, and the expansion and contraction rates of vectors in E^c is dominated by the corresponding rates in E^u and E^s, respectively. Furthermore, we say that f is *center bunched* if the contraction rate on E^s and the expanding rate on E^u are uniformly bounded by the product of the contracting and expanding rates on E^c.

An su-path in M is a concatenation of finitely many subpaths, each of which lies entirely in a single leaf of $\mathcal{W}^s$ or $\mathcal{W}^u$. We say that f is *accessible* if any point in M can be reached from any other along an su-path.

We show that if f is chaotic enough and A is fiber-bunched then the Bochi phenomenon can not occur. That is, (see Chapter 2 for detailed definitions),

Theorem A. *If (f, μ) is a volume preserving partially hyperbolic accessible and center-bunched diffeomorphism over M, and $A : M \to SL(2, \mathbb{R})$ is a Hölder continuous fiber-bunched map with nonvanishing Lyapunov exponents. Then A can not be accumulated by cocycles with zero Lyapunov exponents.*

Moreover, we show that the accessibility assumption in the previous result is necessary.

Theorem B. *There exists a volume preserving partially hyperbolic and center-bunched diffeomorphism f and a Hölder continuous fiber-bunched map A with non-zero Lyapunov exponents which is approximated by cocycles with zero Lyapunov exponents.*

Notice that in the results mention above we consider the Lyapunov exponents as funtions of the cocycle. The purpose of the second part of this work is to study the continuity and semicontinuity of the Lyapunov exponents respect to measures of non-compact support. That is, for the second part of this work we will consider the Lyapunov exponents are functions of the measures. Moreover, we restrict our study to the particular case of *products of random matrices* i,e, when the linear cocycle is given by $A((\alpha_k)_k) = \alpha_0$ where $\alpha_k \in SL(2, \mathbb{R})$ for every $k \in \mathbb{Z}$.

Our main result reads as follows (see Theorem 2.4.3 and Theorem 2.4.4 for a precise statement):

Theorem C. *The function $p \mapsto \lambda_+(p)$ is upper semi-continuous relative to the Wasserstein topology but not with the weak* topology. The same remains valid for $p \mapsto \lambda_-(p)$ with lower semi-continuity.*

The main problem regarding the weak* topology is that it is defined in terms of bounded continuous or bounded Lipschitz functions. That is, we have that $\mu_n \xrightarrow{*} \mu$ if

$$\left| \int \psi d\mu_n - \int \psi \mu \right| \to 0,$$

for all bounded continuous (or Lipschitz) functions (see [21, Chapter 2]). In the case of probability measures with non compact support we don't have a bounded cocycle so we can not guarantee the semicontinuity as in the compact case. However, convergence in the Wasserstein topology, as we will see in Section 2.3, is equivalent to the convergence of integrals of Lipschitz, not necessarily bounded, functions. Moreover, this topology is defined over borel measures with finite first moment and the convergence implies convergence of the moments of order 1. These two properties are the ones that allow us to control the convergence outside a compact set and prove the semicontinuity.

Regarding continuity of Lyapunov exponents we prove the following (see Theorem 2.4.5 for a precise statement):

Theorem D. *The function $p \mapsto \lambda_+(p)$ is not continuous in the Wasserstein Topology. The same remains valid for $p \mapsto \lambda_-(p)$.*

For the proof of this theorem we present an example of measures with vanishing Lyapunov exponent converging to one with strictly positive exponent. The main idea is to take a measure with a countable support containing only hyperbolic matrices and create a sequence by replacing one element of the original support by a rotation that exchanges vertical and horizontal axes. However, the support of the measures constructed this way move further apart from the original support.

The same result can be obtained by considering more variables in the alphabet, for example $SL(2,\mathbb{R})^5$. Hence, instead of one (big) rotation we can decompose it into several small rotations in each coordinate allowing us to leave the supports arbitrarily close.

1.1 Structure of the work

The present work is divided in three parts:

- In Chapter 2 we present basic definitions and some preliminary results in order to give a precise statement of Theorems A through D.

- Chapter 3 is devoted to the proof of the results regarding partially hyperbolic diffeomorphims. Section 3.1 has the preliminary results we are going to need in the proof. The second section, Section 3.2, presents the proof of Theorem A while the proof of Theorem B is presented in Section 3.3.
- In Chapter 4 we focus on the study of probability measures with non compact support. The first section is devoted to the study of semicontinuity while the second one focuses on analyze the continuity.

Definitions and statements

2.1 Linear cocycles and Lyapunov exponents

Let $(M, \mathcal{B}, \mu)$ be a measurable space and f an invertible measure preserving transformation $f\colon (M,\mu) \to (M,\mu)$. A measurable function $A\colon M \to GL(2,\mathbb{R})$ gives the dynamically defined products

$$A^n(x) = \begin{cases} A(f^{n-1}(x)) \dots A(f(x))A(x), & \text{if } n > 0, \\ Id, & \text{if } n = 0, \\ (A^{-n}(f^n(x)))^{-1} = A(f^n(x))^{-1} \dots A(f^{-1}(x))^{-1}, & \text{if } n < 0. \end{cases} \tag{2.1}$$

The **linear cocycle** defined by A over f is the transformation

$$F : M \times \mathbb{R}^2 \to M \times \mathbb{R}^d \quad (x, v) \mapsto (f(x), A(x)v),$$

where its n-th iterate is given by $F^n(x,v) = (f^n(x), A^n(x)v)$.

Let $L^1(\mu)$ denote the space of μ-integrable functions on M and suppose that $\log^+ \|A^{\pm 1}\|$ belongs to $L^1(\mu)$. It follows from the sub-additive ergodic theorem of Kingman [19], that the limits

$$\lambda_+(A,x) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\|,$$
$$\lambda_-(A,x) = \lim_{n\to\infty} \frac{1}{n} \log \|A^{-n}(x)\|^{-1},$$

exist for μ-almost every $x \in M$. We call such limits **Lyapunov exponents**. Moreover, when $\lambda_+(A,x) > \lambda_-(A,x)$ it follows from the well-known theorem of Oseledets [22] that there exists a decomposition $\mathbb{R}^2 = E_x^{u,A} \oplus E_x^{s,A}$ into vector subspaces depending measurably on x such that for μ-almost every point

$$A(x)E_x^{u,A} = E_{f(x)}^{u,A},$$
$$\lambda_+(A,x) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)v\|,$$

for every non-zero $v \in E_x^{u,A}$. Equivalently for s and $\lambda_-(A,x)$. This decomposition is called the *Oseledets decomposition*. Furthermore, the Lyapunov exponents are f-invariant, so if μ is ergodic it implies that they are constant for μ-almost every point x. In this case we write $\lambda_+(A,x) = \lambda_+(A,\mu)$ and $\lambda_-(A,x) = \lambda_-(A,\mu)$.

2.2 Partial hyperbolicity

Let $f : M \to M$ be a C^r, $r \geq 2$, diffeomorphism defined on a compact manifold M, f is said to be **partially hyperbolic** if:

1. There exists a non-trivial splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$ invariant under the derivative Df;

2. There exist a Riemannian metric $\|\cdot\|$ on M, such that we have positive continuous functions ν, $\hat{\nu}$, γ, $\hat{\gamma}$ with ν, $\hat{\nu} < 1$ and $\nu < \gamma < \hat{\gamma}^{-1} < \hat{\nu}^{-1}$ such that, for any unit vector $v \in T_x M$,

$$\begin{aligned} \|Df(x)v\| &< \nu(x) && \text{if } v \in E^s(x), \\ \gamma(x) < \|Df(x)v\| &< \hat{\gamma}(x)^{-1} && \text{if } v \in E^c(x), \\ \hat{\nu}(x)^{-1} < \|Df(x)v\| & && \text{if } v \in E^u(x). \end{aligned}$$

All three sub-bundles E^s, E^c, E^u are assumed to have positive dimension. We say that f is **center-bunched** if

$$\nu < \gamma\hat{\gamma} \text{ and } \hat{\nu} < \gamma\hat{\gamma}.$$

Center bunching means that the hyperbolicity of f dominates the nonconformality of Df on the center. We need this hypothesis because we are going to use some results of [3]. Along this chapter we take M to be endowed with the distance $\operatorname{dist} : M \times M \to \mathbb{R}$ associated to such a Riemannian structure.

The stable and unstable bundles E^s and E^u of a partially hyperbolic diffeomorphism are uniquely integrable [12]. Their integral manifolds form two transverse continuous foliations $\mathcal{W}^s$ and $\mathcal{W}^u$, whose leaves are immersed sub-manifolds of the same class of differentiability as f. These foliations are referred to as the *strong-stable* and *strong-unstable* foliations. They are invariant under f, in the sense that

$$f(\mathcal{W}^s(x)) = \mathcal{W}^s(f(x)) \quad \text{and} \quad f(\mathcal{W}^u(x)) = \mathcal{W}^u(f(x)),$$

where $\mathcal{W}^s(x)$ and $\mathcal{W}^u(x)$ denote the leaves of $\mathcal{W}^s$ and $\mathcal{W}^u$, respectively, passing through any $x \in M$. We say that f is **accessible** if M and $\emptyset$ are the only su-saturated sets. This means that, except of $\emptyset$, M is the only set that is a union of entire strong-stable and strong-unstable leaves.

Let $A : M \to SL(2,\mathbb{R})$ be an α-Hölder continuous map with $0 < \alpha \leq 1$. This means that there exists a constant $C > 0$ such that

$$\|A(x) - A(y)\| \leq C \operatorname{dist}(x,y)^\alpha$$

for all $x, y \in M$ where $\|A\|$ denotes the operator norm of a matrix A, that is,

$$\|A\| = \sup \{\|Av\|/\|v\|;\ \|v\| \neq 0\}.$$

Let $H^\alpha(M)$ denote the space of all such α-Hölder continuous maps. We endow this space with the α-Hölder topology which is generated by the norm

$$\|A\|_\alpha = \sup_{x\in M} \|A(x)\| + \sup_{x\neq y} \frac{\|A(x) - A(y)\|}{\operatorname{dist}(x,y)^\alpha}.$$

We say that the linear cocycle generated by A over f is **fiber-bunched** if

$$\|A(x)\|\|A(x)^{-1}\|\nu(x)^\alpha < 1 \text{ and } \|A(x)\|\|A(x)^{-1}\|\hat{\nu}(x)^\alpha < 1$$

for every $x \in M$. Since our base dynamics f is going to be fixed, we simply say that A is fiber-bunched. Observe that this is an open condition in $H^\alpha(M)$.

2.3 Wasserstein topology

Let (M, μ) and (N, ν) be two probability spaces. *Coupling* μ and ν means constructing a measure π on $M \times N$, such that π projects to μ and ν on the first and second coordinate respectively. When $\mu = \nu$ we call π a *self-coupling*.

If (M, d) is a Polish metric space, for any two probability measures μ, ν on M, the **Wasserstein distance** between μ and ν is defined by the formula

$$W_1(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \int_M d(x,y)d\pi(x,y), \tag{2.2}$$

where the infimum is taken over the set $\Pi(\mu, \nu)$ which denotes the set of all the couplings of μ and ν.

The **Wasserstein space** is the space of probability measures which have a *finite moment of order* 1. By this we mean the space

$$P_1(M) := \left\{\mu \in P(M) : \int_M d(x_0,x)d\mu(x) < +\infty\right\},$$

where $x_0 \in M$ is arbitrary and $P(M)$ denotes the space of Borel probability measures on M. This does not depend on the choice of the point x_0, and W_1 defines a finite distance on it (see [1, Chapter 7]).

An important property of the Wasserstein topology is the **Kantorovich duality**. It establishes that

$$W_1(\mu,\nu) = \sup\left\{\int_M \psi d\mu - \int_M \psi d\nu\right\},$$

where the supremum on the right is over all 1-Lipschitz functions ψ.

The next definition characterizes the convergence in the Wasserstein space $P_1(M)$. From now on the notation $\mu_k \xrightarrow{W} \mu$ means that μ_k converges in the Wasserstein topology, while $\mu_k \xrightarrow{*} \mu$ means that μ_k converges in the weak* topology.

Definition 2.3.1. *[25, Definition 6.8] Let (M, d) be a Polish metric space. Let $(\mu_k)_{k\in\mathbb{N}}$ be a sequence of probability measures in $P_1(M)$ and let μ be another element of $P_1(M)$. Then μ_k is said to converge in the Wasserstein topology to μ, if one of the following equivalent properties is satisfied for some (and then any) $x_0 \in M$:*

1. $\mu_k \xrightarrow{*} \mu$ *and* $\int d(x_0, x)d\mu_k(x) \to \int d(x_0, x)d\mu(x)$;

2. $\mu_k \xrightarrow{*} \mu$ *and*

$$\limsup_{k\to\infty} \int d(x_0, x)d\mu_k(x) \leq \int d(x_0, x)d\mu(x);$$

3. $\mu_k \xrightarrow{*} \mu$ *and*

$$\lim_{R\to\infty} \limsup_{k\to\infty} \int_{d(x_0,x)\geq R} d(x_0, x)d\mu_k(x) = 0;$$

4. *For all continuous functions φ with $|\varphi(x)| \leq C(1 + d(x_0, x))$, $C \in \mathbb{R}$, one has*

$$\int \varphi(x)d\mu_k(x) \to \int \varphi(x)d\mu(x).$$

A crucial fact is that the Wasserstein distance W_1 metrizes the convergence in the Wasserstein topology in $P_1(M)$. In other words, $\mu_k \xrightarrow{W} \mu$ if and only if $W_1(\mu_k, \mu) \to 0$. This equivalence also implies that W_1 is continuous on $P_1(M)$ (see [25, Theorem 6.18]).

Theorem 2.3.1 (Topology in $P_1(M)$)**.** *Let (M, d) be a Polish metric space. Then the Wasserstein distance W_1, metrizes the convergence in the Wasserstein topology in the space $P_1(M)$. Moreover, with this metric $P_1(M)$ is also a complete separable metric space and, any probability measure can be approximated by a sequence of probability measures with finite support.*

2.4 Main results

With the definitions mention in the last sections we now proceed to properly state the results mention in Chapter 1.

2.4.1 Partially hyperbolic base dynamics

Let $f : M \to M$ be a partially hyperbolic diffeomorphism on a compact manifold M and μ a probability measure in the *Lebesgue class* of M. By Lebesgue class we mean the set of measures that are generated by a volume form. Moreover, we say that f is volume preserving if it preserves some probability measure in the Lebesgue class of M.

The main results regarding partially hyperbolic diffeomorphisms are the following.

Theorem 2.4.1. *Let $f : M \to M$ be a C^r, $r \geq 2$, partially hyperbolic, volume preserving, center bunched and accessible diffeomorphism defined on a compact manifold M. Also, let μ be an ergodic f-invariant measure in the Lebesgue class. If $A \in H^\alpha(M)$ is fiber-bunched and $\lambda^u(A, \mu) > \lambda^s(A, \mu)$ then A can not be accumulated by cocycles with zero Lyapunov exponents.*

We observe that a similar result can be stated in terms of $GL(2, \mathbb{R})$-valued cocycles changing 'cocycles with zero Lyapunov exponents' by 'cocycles with just one Lyapunov exponent'.

Indeed, the accessibility property guarantees connectedness of M. Hence, by continuity of A either, $\det(A(x)) > 0$ for every $x \in M$ or $\det(A(x)) < 0$ for every $x \in M$. Suppose we are in the first case (the other case can be easily deduced from this one). Then, given $A : M \to GL(2, \mathbb{R})$ consider $g_A : M \to \mathbb{R}$ the map defined by $g_A(x) = (\det A(x))^{\frac{1}{2}}$ and $B : M \to SL(2, \mathbb{R})$ such that $A(x) = g_A(x)B(x)$. Therefore,

$$\lambda^{u/s}(A, \mu) = \lambda^{u/s}(B, \mu) + \int \log(g_A(x)) \, d\mu(x),$$

and consequently,

$$\lambda^u(A, \mu) = \lambda^s(A, \mu) \Longleftrightarrow \lambda^u(B, \mu) = 0 = \lambda^s(B, \mu).$$

We also present an example showing that the accessibility assumption in the previous theorem is necessary. More precisely,

Theorem 2.4.2. *There exists a volume preserving partially hyperbolic and center-bunched diffeomorphism f and a Hölder continuous fiber-bunched map A with non-zero Lyapunov exponents which is approximated by cocycles with zero Lyapunov exponents.*

2.4.2 Measures with non compact support

Let $M = SL(2,\mathbb{R})^{\mathbb{Z}}$ and, let $f : M \to M$ be the shift map over M defined by

$$(\alpha_n)_n \mapsto (\alpha_{n+1})_n.$$

Consider the function

$$A : M \to SL(2,\mathbb{R}), \quad (\alpha_n)_n \mapsto \alpha_0,$$

and we define its n-th iterate, the *product of random matrices,* by

$$A^n((\alpha_k)_k) = \alpha_{n-1} \cdots \alpha_0.$$

Given an invariant measure p in $SL(2,\mathbb{R})$ we can define $\mu = p^{\mathbb{Z}}$ which is an invariant measure in M.

It is a well-known fact that when the measures have compact support, the Lyapunov exponents are semicontinuous with the weak* topology (see for example [24, Chapter 9]). However, in the non compact setting this is no longer true. If they were semicontinuos then every measure with vanishing Lyapunov exponents would be a point of continuity. The next theorem shows that this is not the case.

Theorem 2.4.3. *There exist a measure p and a sequence of measures $(q_n)_n$ on $SL(2,\mathbb{R})$ converging to q in the weak* topology, such that $\lambda_+(q_n) \geq 1$ for n large enough but $\lambda_+(q) = 0$.*

Consider in $SL(2,\mathbb{R})$ the metric given by

$$d(\alpha,\beta) = \|\alpha - \beta\| + \|\alpha^{-1} - \beta^{-1}\|.$$

Since the space $SL(2,\mathbb{R})$ is a Polish metric space with this metric we can consider the Wasserstein topology in $P_1(SL(2,\mathbb{R}))$.

The Wasserstein topology is stronger than the weak* topology, as mentioned in Definition 2.3.1. The principal consequence of the convergence in the Wasserstein topology is that it implies convergence of the moments of order 1. This allow us to control the weight of integrals outside compact sets and, proof semi-continuity of the Lyapunov exponents in $P_1(SL(2,\mathbb{R}))$. We are thus led to the following result.

Theorem 2.4.4. *The function defined on $P_1(SL(2,\mathbb{R}))$ by $p \to \lambda_+(p)$ is upper semi-continuous relative to the Wasserstein topology. The same remains valid for the function $p \to \lambda_-(p)$ with lower semi-continuity.*

Finally, we will present a construction of discontinuity points of the Lyapunov exponents as functions of the probability measure, relative to the Wasserstein topology.

This implies that the Wasserstein topology is not enough to guarantee continuity of the Lyapunov exponents.

Theorem 2.4.5. *There exist a measure q and a sequence of measures $(q_n)_n$ on $SL(2,\mathbb{R})$ converging to q in the Wasserstein topology, such that $\lambda_+(q_n) = 0$ for all $n \in \mathbb{N}$ but $\lambda_+(q) > 0$.*

Partially hyperbolic base dynamics

The aim of this chapter is to give a proof for Theorem A and Theorem B. Let us give an outline of the proof of Theorem A, it goes by contradiction. Assume there exist a sequence $\{A_k\}_k$ with $\lambda^u(A_k, \mu) = \lambda^s(A_k, \mu) = 0$ for every $k \in \mathbb{N}$ and such that A_k converges to A in the Hölder topology. The basic strategy is to consider the projective cocycles $F_{A_k}, F_A : M \times \mathbb{P}^1 \to M \times \mathbb{P}^1$ defined by (A, f) and (A_k, f) respectively, and to analyze the probability measures m and m_k on $M \times \mathbb{P}^1$ that are invariant under F_A and F_{A_k} respectively, and project down to μ on M.

The accessibility condition allow us to define, in Section 3.1.2, holonomy maps H^A_γ for every su-path γ on every point $x \in M$. Using a result of Avila, Santamaria, Viana in [3] we show in Section 3.2.1 that the Oseledets decomposition is invariant by those holonomy maps. Thus, we can separate our proof in two cases:

In the first case we assume there exist $x \in M$ and a non trivial su-loop γ on x such that H^A_γ is hyperbolic. Hence, for k sufficiently large the holonomy maps $H^{A_k}_\gamma$ are also hyperbolic and the conditional measures m^k_x have at most two atoms. In Section 3.2.1 we prove uniform convergence of the conditional measures and, in Section 3.2.2 we use it to prove that m^k_x can not have a finite number of atoms contradicting this case.

In the second case we assume that for every point x and every su-loop γ at x that $H^A_\gamma = \mathrm{id}$. Section 3.1.3 gives a characterization of the sequence of holonomy maps for the cocycles A_k. This characterization give us again two cases: either there exist a nontrivial su-loop γ at some point x such that $H^{A_k}_\gamma$ is hyperbolic for infinitely many k or they are the identity for every su-loop for some k large enough. The hyperbolic case is solved as before, for the identity case, we can perform a change of coordinates that makes the cocycles A and A_k constant without changing its Lyapunov exponents, this is explained in Section 3.1.2. The proof concludes by establishing the convergence of the Lyapunov exponents for the constant cocycles. This is done in Section 3.2.3.

Section 3.3 contains the proof of Theorem B as well as another construction showing that we have a fiber-bunched cocycle A over a partially hyperbolic and center-bunched map f with arbitrarily large Lyapunov exponent λ_+ which can be approximated by cocycles with zero Lyapunov exponents.

The results presented in this chapter are the product of a joint work with Lucas Backes and Mauricio Poletti in [6].

3.1 Preliminary results

In this section we recall some classical notions and present some useful results that are going to be used in the proof of our main theorem. Along this chapter we consider $f : M \to M$, $A \in H^\alpha(M)$ and μ be as in Theorem A.

3.1.1 Holonomies and disintegrations

Given $x, y \in M$, we define the equivalence relation $x \sim^s y$ by $y \in \mathcal{W}^s(x)$. Observe that this is f-invariant, that is, if $x \sim^s y$ then $f(x) \sim^s f(y)$. Given $L > 0$, we write $x \sim^s_L y$ when there exist a sequence of points $x = z_0, ..., z_n = y$ such that $z_i \sim^s z_{i+1}$ and $d_{\mathcal{W}^s}(z_{i+1}, z_i) \leq L$ for every $i = 1, \dots, n-1$.

For every pair of points $x, y \in M$ satisfying that $x \sim^s y$, the fiber-bunched assumption assures that the limit

$$H^{s,A}_{xy} = \lim_{n \to +\infty} A^n(y)^{-1} \circ A^n(x)$$

exists (see [3, Proposition 3.2]). Moreover, by [3, Remark 3.4] we know that for every $L > 0$, the map

$$(x, y, A) \to H^{s,A}_{xy}$$

is continuous on $\mathcal{W}^s_L \times H^\alpha(M)$, where $\mathcal{W}^s_L = \{(x, y) \in M \times M; x \sim^s_L y\}$.

The family of maps $H^{s,A}_{xy}$ is called *an stable holonomy* for the cocycle (A, f). Also by [3, Proposition 3.2] we have for $x \sim^s_L y$ and $z \sim^s_L y$,

- $H^{s,A}_{xx} = \mathrm{id}$,
- $H^{s,A}_{xy} = H^{s,A}_{zy} \circ H^{s,A}_{xz}$,
- $\|H^A_{xy}\| \leq N$ for some $N > 0$ and,
- $H^{s,A}_{f^j(x)f^j(y)} = A^j(y) H^{s,A}_{xy} A^j(x)^{-1} \quad \forall j \geq 0.$

Dually, we write $x \sim^u y$ if $y \in \mathcal{W}^u(x)$ and, we define the *unstable holonomy* $H^{u,A}_{xy}$ as the stable holonomies for (A^{-1}, f^{-1}). That is

$$H^{u,A}_{xy} = \lim_{n \to -\infty} A^n(y)^{-1} \circ A^n(x)$$

whenever x and y are on the same strong-unstable leaf. Even more, $(x, y, A) \to H^{u,A}_{xy}$ is continuous on $\mathcal{W}^u_L \times H^\alpha(M)$, where $\mathcal{W}^u_L = \{(x, y) \in M \times M; x \sim^u_L y\}$.

We say that a measure m on $M \times \mathbb{P}^1$ *projects* on μ if $\pi_* m = \mu$ where π is the canonical projection $\pi : M \times \mathbb{P}^1 \to M$. By the Ergodic Decomposition theorem [21, Theorem 5.1.3] any such measure admits a *disintegration* with respect to the partition $\{\{x\} \times \mathbb{P}^1\}_{x \in M}$ and the measure μ, that is, there exists a family of measures $\{m_x\}_{x \in M}$ on $\{\{x\} \times \mathbb{P}^1\}_{x \in M}$ so that for every measurable $B \subset M \times \mathbb{P}^1$,

- The map $x \to m_x(B)$ is measurable,
- $m_x(\{x\} \times \mathbb{P}^1) = 1$ and,
- $m(B) = \int_M m_x(B \cap (\{x\} \times \mathbb{P}^1)) d\mu(x)$.

Moreover, such disintegrantion is essentially unique [23]. Identifying each fiber $\{x\} \times \mathbb{P}^1$ with $\mathbb{P}^1$, we can think of $x \to m_x$ as a map from M to the space of probability measures on $\mathbb{P}^1$ endowed with the weak* topology.

Given $B \in GL(2, \mathbb{R})$ we write $\mathbb{P}B : \mathbb{P}^1 \to \mathbb{P}^1$ for the induced projective map. That is, the map defined by $\mathbb{P}B[v] = [Bv]$.

The *projective cocycle* associated to A is the map $F_A : M \times \mathbb{P}^1 \to M \times \mathbb{P}^1$ given by

$$F_A(x, v) = (f(x), \mathbb{P}A(x)[v]).$$

Observe that m is F_A-invariant if and only if $(\mathbb{P}A(x))_* m_x = m_{f(x)}$ for μ-almost every point $x \in M$.

We say that a F_A-invariant measure m projecting on μ is *essentially s-invariant* if there exists a total measure set $M^s \subset M$ such that for every $x, y \in M^s$ satisfying $x \sim^s y$ we have

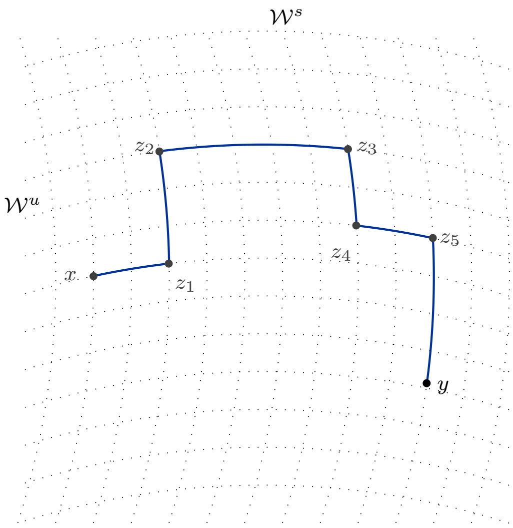
$$\left(H^{s,A}_{xy}\right)_* m_x = m_y.$$

Such measure m is also known as an *s-state*. One speaks of s-invariant if $M^s = M$. Analogously, we say that m is *essentially u-invariant* (or an *u-state*) if the same is true replacing stable by unstable in the previous definition. We say that m is *essentially su-invariant* if it is simultaneously essentially s-invariant and essentially u-invariant. The main property of essentially su-invariant measures is the following

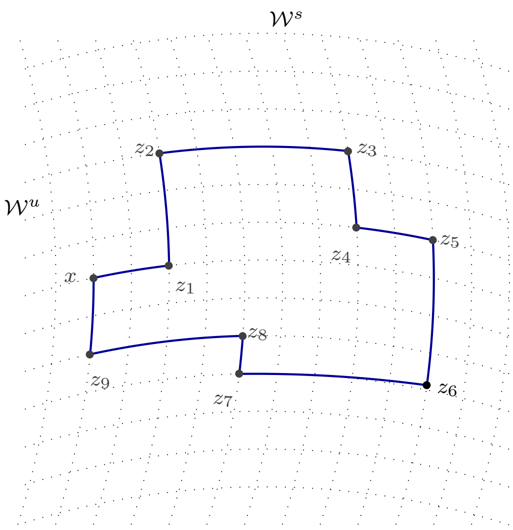
Proposition 3.1.1. *[3, Theorem B] If $\lambda_+(A, \mu) = \lambda_-(A, \mu)$, any F_A-invariant measure m projecting on μ admits a disintegration $\{m_x\}_{x \in M}$ for which $M^s = M^u = M$. Moreover, accessibility of f implies that m_x depends continuously on the base point $x \in M$ in the weak* topology.*

3.1.2 Accessibility and holonomies

An *su-path* from x to y is a path connecting x and y which is a concatenation of finitely many subpaths, each of which lies entirely in a single leaf of $\mathcal{W}^s$ or a single leaf of $\mathcal{W}^u$. Every sequence of points $x = z_0, z_1, \ldots, z_n = y$, with the property that z_i and z_{i+1} lie in the same leaf of either W^s or W^u , for $i = 0, ..., n-1$ defines a unique su-path (see Figure 3.1). If in addition $x = y$, then the sequence determines an *su-loop* or a *closed su-path* (see Figure 3.2). With these terminology the accessibility of f means that any point in M can be reached from any other along an su-path.



su-path



su-loop

We define the concatenation of two su-paths γ_1 given by $z_0, \ldots, z_n$ and γ_2 given by $z'_0, z'_1, \ldots, z'_m$, with $z'_0 = z_n$, by $\gamma_1 \wedge \gamma_2$ which is the su-path given by the sequence of points $z_0, \ldots, z_n, z'_1, \ldots, z'_m$.

We say that an su-path γ defined by the sequence $x = z_0, z_1, \ldots, z_n = y$ is a (K, L)-*path* if $n \leq K$ and $d_{\mathcal{W}^*}(z_{i+1}, z_i) \leq L$ for every $i = 1, \ldots, n-1$ where $d_{\mathcal{W}^*}$ is the distance induced by the Riemannian structure on the submanifold $\mathcal{W}^*$ for $* = s, u$. Observe that, by the compactness of M and continuity of stable (unstable) manifolds of bounded size, the space of (K, L)-paths is compact. In particular, Wilkinson [27] proved that accessibility implies uniform accessibility:

Lemma 3.1.2. *[27, Lemma 4.5] There exist constants K and L such that every pair of points in M can be connected by an (K, L)-path.*

Consider K and L given by the lemma above. If γ is the su-path defined by the sequence $z_0, z_1, \ldots, z_n$ then we write $H^A_\gamma = H^{*,A}_{z_{n-1}z_n} \circ \ldots \circ H^{*,A}_{z_0 z_1}$ for $* \in \{s, u\}$.

Let us assume that $H^A_\gamma = \mathrm{id}$ for every $(3K, L)$-loop. This implies that the same remains valid for *every* su-loop. Indeed, observe initially that any su-loop γ can be transformed into an su-loop with legs of size at most L just by breaking one "large" leg into several with smaller sizes. Thus, it is enough to consider su-loops with legs of size at most L.

If γ is a $(2K, L)$-path from x to y then, by Lemma 3.1.2, there exists a (K, L)-path γ' from x to y. If $-\gamma'$ denotes the path γ' with opposite orientation then $\gamma \wedge (-\gamma')$ is a $(3K, L)$-loop and

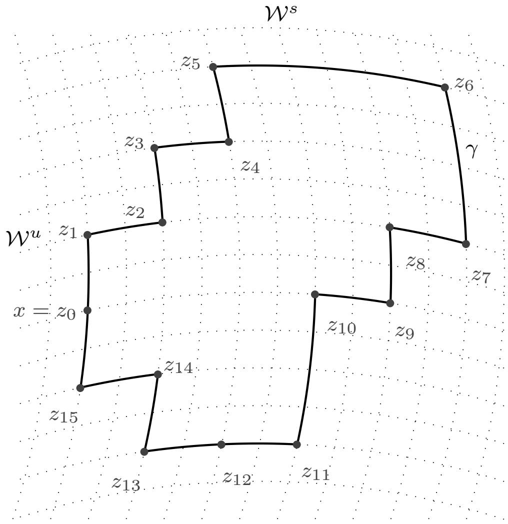
$$H^A_\gamma \circ (H^A_{\gamma'})^{-1} = H^A_\gamma \circ H^A_{-\gamma'} = H^A_{\gamma \wedge (-\gamma')} = \mathrm{id}\,.$$

Hence, $H^A_\gamma = H^A_{\gamma'}$.

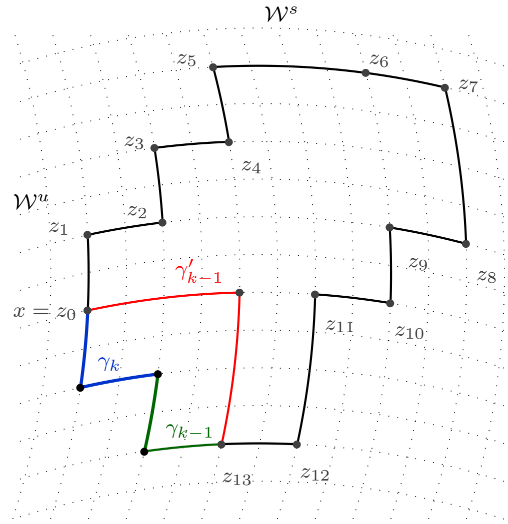
Now, taking any su-loop γ with an arbitrary number of legs whose lengths are at most L we can decompose it as $\gamma = \gamma_1 \wedge \cdots \wedge \gamma_k$, where every γ_i is a (K, L)-path. In particular, $\gamma_{k-1} \wedge \gamma_k$ is a $(2K, L)$-path and by the previous argument we can replace it by a (K, L)-path γ'_{k-1} with the same starting and ending points and, so that

$$H^A_{\gamma_{k-1} \wedge \gamma_k} = H^A_{\gamma'_{k-1}}.$$

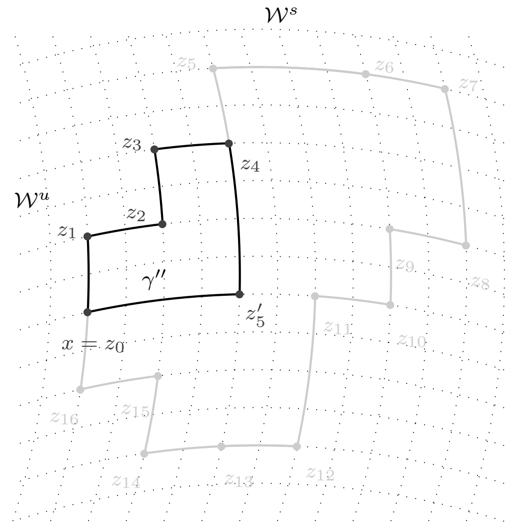
Thus, taking $\gamma' = \gamma_1 \wedge \cdots \wedge \gamma_{k-2} \wedge \gamma'_{k-1}$ we have that γ and γ' have the same starting and ending points and $H^A_\gamma = H^A_{\gamma'}$. Repeating this procedure a finite number of times we get some (K, L)-loop γ'' such that $H^A_\gamma = H^A_{\gamma''} = \mathrm{id}$. Finally, we conclude that $H^A_\gamma = \mathrm{id}$ for every su-loop proving our claim. An example of this process is shown in Figure 3.3 where $L = 4$ and $K = 2$.



Original su-loop γ



Construction of γ'_{k-1}



Final su-loop γ''

Process to transform an arbitrary su-loop into a $(3K, L)$-loop.

As a consequence we get that if γ is an su-path connecting x and y then H_γ^A *does not depend* on γ. In fact, if γ_1 and γ_2 are su-paths connecting x and y then $\gamma_1 \wedge (-\gamma_2)$ is an su-loop and thus

$$H_{\gamma_1}^A \circ (H_{\gamma_2}^A)^{-1} = H_{\gamma_1}^A \circ H_{-\gamma_2}^A = H_{\gamma_1 \wedge (-\gamma_2)}^A = \mathrm{id}\,.$$

Let us denote this common value simply by H_{xy}^A. From the properties of the holonomies and the fact that any two points $x, y \in M$ can be connected by a (K, L)-path it follows that

- $H_{yz}^A H_{xy}^A = H_{xz}^A$,
- $A(y) H_{xy}^A = H_{f(x)f(y)}^A A(x)$,
- $A \to H_{xy}^A$ is uniformly continuous for any pair of points $x, y \in M$ and
- $\|H_{xy}^A\| \le N$ for some $N > 0$ and any $x, y \in M$.

Fix $x \in M$ and, given $y \in M$, consider the following transformation

$$\hat{A}(y) = H_{f(y)x}^A A(y) H_{xy}^A. \tag{3.1}$$

We are going to prove that this change of coordinates makes the cocycle (A, f) constant without changing its Lyapunov exponents.

Notice that by the properties of the holonomies

$$\hat{A}^2(y) = \hat{A}(f(y))\hat{A}(y) = H_{f^2(y)x}^A A(f(y)) H_{xf(y)}^A H_{f(y)x}^A A(y) H_{xy}^A,$$

and consequently $\hat{A}^2(y) = H_{f^2(y)x}^A A^2(y) H_{xy}^A$. More generally, $\hat{A}^n(y) = H_{f^n(y)x}^A A^n(y) H_{xy}^A$ for every $n \in \mathbb{N}$. Hence, $(\hat{A}, f)$ and (A, f) have the same Lyapunov exponents. Indeed, notice that

$$\begin{aligned}
\lambda_+(\hat{A}, \mu) &= \lim_{n\to\infty} \frac{1}{n} \log \|\hat{A}^n(y)\| \\
&= \lim_{n\to\infty} \frac{1}{n} \log \|H_{f^n(y)x}^A A^n(y) H_{xy}^A\| \\
&\le \lim_{n} \frac{1}{n} (2 \log N + \log \|A^n(y)\|) \\
&= \lambda_+(A, \mu).
\end{aligned}$$

Similarly, since $A^n(y) = H_{xf^n(y)}^A \hat{A}^n(y) H_{yx}^A$ we have that $\lambda_+(A, \mu) \le \lambda_+(\hat{A}, \mu)$. Thus, we proved our claim.

Moreover, for any $z, y \in M$,

$$\begin{aligned}
\hat{A}(z)^{-1}\hat{A}(y) &= \left(H^A_{f(z)x}A(z)H^A_{xz}\right)^{-1} H^A_{f(y)x}A(y)H^A_{xy} \\
&= H^A_{zx}A(z)^{-1}H^A_{xf(z)}H^A_{f(y)x}A(y)H^A_{xy} \\
&= H^A_{zx}A(z)^{-1}H^A_{f(y)f(z)}A(y)H^A_{xy} \\
&= H^A_{zx}A(z)^{-1}A(z)H^A_{yz}H^A_{xy} \\
&= H^A_{zx}H^A_{yz}H^A_{xy} \\
&= H^A_{zx}H^A_{xz} \\
&= \mathrm{id}\,.
\end{aligned}$$

In particular, $\hat{A}$ is constant and consequently its largest Lyapunov exponent is the logarithm of the norm of the greatest eigenvalue of $\hat{A}$. Summarizing, if $H^A_\gamma = \mathrm{id}$ for every $(3K, L)$-loop γ then we can perform a change of coordinates that makes the cocycle (A, f) constant without changing its Lyapunov exponents.

3.1.3 $PSL(2,\mathbb{R})$ cocycles and invariant measures in $\mathbb{P}^1$

Let $PGL(2,\mathbb{R})$ be the projective linear group, that is the induced action of the general linear group on the associated projective space $\mathbb{P}^1$. Let $\pi : \mathbb{R}^2 \setminus \{0\} \to \mathbb{P}^1$ be the natural projection given by $v \mapsto [v]$. Each automorphism $\tilde{P} \in GL(2,\mathbb{R})$ induces a projective transformation $P \in PGL(2,\mathbb{R})$ through $P[v] = [\tilde{P}v]$.

On the other hand, endomorphisms of $\mathbb{R}^2$ (i.e. linear maps) do not project, in general, to self maps of $\mathbb{P}^1$. Nevertheless, it was pointed out by Furstenberg [15] that the space of projective maps has a natural compactification. If $\tilde{Q} \in \mathrm{End}(\mathbb{R}^2)$ is a linear transformation of rank $r > 0$ with kernel $\ker \tilde{Q}$ and image $\operatorname{Im} \tilde{Q}$ then, $\tilde{Q}$ determines a *quasi-projective transformation* Q of $\mathbb{P}^1$ given by $Q([v]) = [\tilde{Q}(v_1)]$ where v_1 is any vector such that $v - v_1 \in \ker(\tilde{Q})$. Observe that Q is defined and continuous on the complement of the projective subspace $\ker Q = \{[v] : v \in \ker \tilde{Q}\}$, and its image is $\operatorname{Im} Q = \{[v] : v \in \operatorname{Im} \tilde{Q}\}$. The number r is called the rank of this quasiprojective transformation. Rank 1 quasi-projective transformations are *quasi-constant maps*, each of them is undefined on a hyperplane in $\mathbb{P}^1$ and its image is a single point. We refer the reader to [17], [13] or [10] for a deeper discussion of this topic.

The space of quasi-projective transformations inherits a topology from the space of linear maps, through the natural projection $\tilde{\pi} : Q \mapsto \tilde{Q}$. Clearly, every quasi-projective transformation Q is induced by some linear map $\tilde{Q}$ such that $\|\tilde{Q}\| = 1$. It follow that the space of quasi-projective transformations is compact for this topology (see [13, Theorem 2.83]). In particular, every sequence of projective transformations has a subsequence converging to some quasi-projective transformation Q.

The next result will be needed in the proof of the proposition below. Its proof can be found in [10, Lemma 6.1] for projective transformations in $\mathbb{CP}^1$. The notion of quasi-projective maps has been extended to transformations on Grassmannian manifolds by Gol'dsheid and Margulis [16] and a proof of the following lemma in this context can be found in [2, Lemma 2.4].

Lemma 3.1.3. *If $(P_n)_n$ is a sequence of projective transformations converging to a quasi-projective transformation Q, and $(\nu_n)_n$ is a sequence of probability measures in $\mathbb{P}^1$ weakly converging to some probability ν_0 with $\nu_0(\ker Q) = 0$ then $(P_n)_*\nu_n$ converges weakly to $Q_*\nu_0$.*

The following result plays an important part in our proof of Theorem A below.

Proposition 3.1.4. *Let $(L_n)_n$ be a sequence of $SL(2,\mathbb{R})$ matrices converging to* id *and, for each $n \in \mathbb{N}$ let η_n be an L_n-invariant measure on $\mathbb{P}^1$ converging in the weak* topology to $\frac{1}{2}(\delta_p + \delta_q)$ for some $p, q \in \mathbb{P}^1$ with $p \neq q$. Then for every n sufficiently large either L_n is hyperbolic or $L_n = \mathrm{id}$.*

Proof. The proof is by contradiction. We start observing that as L_n converges to the identity and the trace map is continuous, all the matrices have positive trace for n sufficiently large. Consequently, if L_n is not the identity we have three possibilities: the trace $tr(L_n) > 2$ which means that the matrix L_n is hyperbolic, $tr(L_n) < 2$ so the matrix L_n is elliptic or $tr(L_n) = 2$ and the matrix L_n is parabolic and is non diagonalizable with both eigenvalues equal to 1.

Elliptic case: Suppose initially that all the matrices L_n have $tr(L_n) < 2$. Hence it is conjugated to a rotation of angle $\theta_n = \arccos\left(\frac{tr(L_n)}{2}\right)$. In particular, for each $n \in \mathbb{N}$ there exists $P_n \in SL(2,\mathbb{R})$ so that $L_n = P_n^{-1} R_{\theta_n} P_n$ where R_{θ_n} stands for the rotation of angle θ_n. Moreover, since $tr(L_n) \xrightarrow{n\to+\infty} 2$, we get that $\theta_n \xrightarrow{n\to+\infty} 0$.

Now, for each $n \in \mathbb{N}$ let us consider $\nu_n = P_{n*}\eta_n$ which is an R_{θ_n}-invariant measure. In fact, for every measurable set $A \subset \mathbb{P}^1$

$$\begin{aligned}
\nu_n(R_{\theta_n}^{-1}A) &= \eta_n((P_n^{-1} \circ R_{\theta_n})A) \\
&= \eta_n((R_{\theta_n} \circ P_n)^{-1}A) \\
&= \eta_n((P_n \circ L_n)^{-1}A) \\
&= \eta_n((L_n^{-1} \circ P_n^{-1})A) \\
&= \eta_n(P_n^{-1}A) \\
&= \nu_n(A).
\end{aligned}$$

We start observing that there exists a subsequence $\{n_j\}_j$ so that ν_{n_j} converges to Leb where Leb stands for the Lebesgue measure on $\mathbb{P}^1$. Indeed, if θ_n is an irrational number then we know that the only R_{θ_n}-invariant measure is Leb. In particular, $\nu_n =$ Leb. Thus, if there are infinitely many values of n for which θ_n is an irrational number we are done.

Suppose then that θ_n is a rational number for every $n \in \mathbb{N}$. In particular, R_{θ_n} is periodic and denoting by q_n its period, since $\theta_n \xrightarrow{n\to+\infty} 0$, we have that $q_n \xrightarrow{n\to+\infty} +\infty$.

In what follows we make an abuse of notation thinking of $\mathbb{P}^1$ as $[0,1]$ identifying the extremes of the interval.

Let $\varphi : \mathbb{P}^1 \to \mathbb{R}$ be a continuous map and $\varepsilon > 0$. Since $\mathbb{P}^1$ is compact, there exists $\delta > 0$ so that $\mid \varphi(x) - \varphi(y) \mid < \varepsilon$ whenever $d(x,y) < \delta$. Thus, taking $n \gg 0$ so that $q_n > \frac{1}{\delta}$ we get that $\mid \varphi(x) - \varphi\left(\frac{j}{q_n}\right) \mid < \varepsilon$ for every $x \in I_j = \left[\frac{j}{q_n}, \frac{j+1}{q_n}\right)$ and $j = 0, 1, \ldots, q_n - 1$. In particular,

$$\left| \frac{1}{\nu_n\left(\left[\frac{j}{q_n}, \frac{j+1}{q_n}\right)\right)} \int_{\frac{j}{q_n}}^{\frac{j+1}{q_n}} \varphi d\nu_n - \varphi\left(\frac{j}{q_n}\right) \right| < \varepsilon. \tag{3.2}$$

Since ν_n is R_{θ_n}-invariant and

$$I = \bigcup_{j=0}^{q_n-1} I_j = \bigcup_{j=0}^{q_n-1} R_{\theta_n}^{-j}(I_0),$$

then $\nu_n\left(\left[\frac{j}{q_n}, \frac{j+1}{q_n}\right)\right) = \frac{1}{q_n}$ for every $j = 0, 1, \ldots, q_n - 1$. Summing the expression in 3.2 for j from 0 up to $q_n - 1$ and dividing both sides by q_n we get that

$$\left| \int_0^1 \varphi d\nu_n - \frac{1}{q_n} \sum_{j=0}^{q_n-1} \varphi\left(\frac{j}{q_n}\right) \right| < \varepsilon.$$

On the other hand, since φ is Riemann integrable,

$$\lim_{n\to\infty} \frac{1}{q_n} \sum_{j=0}^{q_n-1} \varphi\left(\frac{j}{q_n}\right) = \int \varphi d\text{Leb}$$

which implies that $\nu_n \xrightarrow{n\to+\infty}$ Leb as claimed. So, restricting to a subsequence, if necessary, we may assume that $\nu_n \xrightarrow{n\to+\infty}$ Leb.

We now analyze the accumulation points of $\eta_n = {P_n^{-1}}_*\nu_n$. If $\{P_n^{-1}\}_n$ stay in a compact set of $SL(2,\mathbb{R})$ then, taking a subsequence if necessary, we may assume that there exists $P \in SL(2,\mathbb{R})$ so that $P_n^{-1} \to P$. In particular, $\lim_{n\to\infty} \eta_n = P_*\text{Leb}$ which contradicts our assumption since P_*Leb is non-atomic. If $\|P_n^{-1}\| \to \infty$

then we can work on the compactification of quasi-projective transformations. In particular, restricting to a subsequence, if necessary, we may suppose that $Q_n = P_n^{-1}/\|P_n^{-1}\|$ converges to some quasi-projective map Q. Note that, since $\|P_n^{-1}\| = \|P_n\| \to \infty$, $\|Q_n\| = 1$ and $|\det Q_n| = \lim_n 1/\|P_n\|^2 = 0$, then Q has rank 1 i.e. is a quasi-constant map and its image is a single point z (see for instant the proof of [24, Lemma 6.4]).

Thus, as the kernel has zero Lebesgue measure we can apply Lemma 3.1.3 to conclude that

$$\lim_{n\to\infty} {P_n^{-1}}_*\nu_n = \lim_{n\to\infty} Q_{n*}\nu_n = Q_*\mathrm{Leb} = \delta_z$$

which is a contradiction. Consequently, L_n may be elliptic only for finitely many values of n.

Parabolic case: To conclude the proof it remains to rule out the cases when $tr(L_n) = 2$ and the matrix are non diagonalizable for infinitely many values of n. So, suppose L_n is non diagonalizable and both of its eigenvalues are 1 for every n. Then by the Jordan's normal decomposition we have

$$L_n = P_n^{-1}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} P_n$$

for some $P_n \in GL(2,\mathbb{R})$. Consequently, the only invariant measure for L_n is atomic and have only one atom, contradicting the fact that $\eta_n \xrightarrow{n\to+\infty} \frac{1}{2}(\delta_p + \delta_q)$. Thus, L_n can be parabolic and different from id only for finitely many values of n concluding the proof of the proposition. □

Let us consider the *projective special linear group* given by $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm Id\}$. That is, given $A, B \in SL(2,\mathbb{R})$ let us define equivalence relation $\sim$ given by $A \sim B$ if and only if $A = B$ or $A = -B$. Given $A \in SL(2,\mathbb{R})$, let

$$[[A]] = \{B \in SL(2,\mathbb{R}); B \sim A\}$$

be the equivalence class of A with respect to $\sim$. Then, we have that $PSL(2,\mathbb{R}) = \{[[A]]; A \in SL(2,\mathbb{R})\}$. Observe that the norm $\|\cdot\|$ on $SL(2,\mathbb{R})$ naturally induces a norm, which we are going to denote by the same symbol, on $PSL(2,\mathbb{R})$: given $A \in SL(2,\mathbb{R})$,

$$\|[[A]]\| := \|A\| = \|-A\|.$$

Given $A : M \to SL(2,\mathbb{R})$ let us consider $\tilde{A} : M \to PSL(2,\mathbb{R})$ defined by $\tilde{A}(x) = [[A(x)]]$. By Kingman's subadditive ergodic theorem [19] and the ergodicity of μ it follows that the limit

$$L(\tilde{A},\mu) = \lim_{n\to+\infty} \frac{1}{n}\log\|\tilde{A}^n(x)\|$$

exists and is constant for μ-almost every $x \in M$. In particular, since $\tilde{A}^n(x) = [[A^n(x)]]$ then $\|A^n(x)\| = \|\tilde{A}^n(x)\|$ for every $x \in M$ and $n \in \mathbb{N}$. Hence, we get that $\lambda^u(A,\mu) = L(\tilde{A},\mu)$. Another simple observation is that for every $v \in \mathbb{P}^1$, $[A(x)v] = [\tilde{A}(x)v]$ and, consequently, the action induced by A on $\mathbb{P}^1$ coincide with the action of $\tilde{A}$ on $\mathbb{P}^1$.

Moreover, $H^{\tilde{A}}_\gamma = [[H^A_\gamma]] \in PSL(2,\mathbb{R})$ is well defined and have similar properties with respect to $\tilde{A}$ as those of H^A_γ with respect to A described in Section 3.1.2. In particular, a similar conclusion to that of Section 3.1.2 holds for $\tilde{A}$ whenever $H^{\tilde{A}}_\gamma = [[\mathrm{id}]]$ for every $(3K,L)$-loop γ: we can perform a change of coordinates that makes the cocycle $(\tilde{A},f)$ constant without changing $L(\tilde{A},\mu)$. Consequently, denoting this new cocycle by $\hat{\tilde{A}}$, it follows that $L(\tilde{A},\mu)$ is equal to logarithm of the norm of the greatest eigenvalue of any representative of $\hat{\tilde{A}}$.

Furthermore, Proposition 3.1.4 also have a counterpart for $PSL(2,\mathbb{R})$ cocycles. In order to state it, recall that a sequence $\{\tilde{L}_n\}_n$ in $PSL(2,\mathbb{R})$ is said to converge to $\tilde{L} \in PSL(2,\mathbb{R})$ if there are representatives L and L_n in $SL(2,\mathbb{R})$ of $\tilde{L}$ and $\tilde{L}_n$, respectively, so that the sequence $\{L_n\}_n$ converges to L in $SL(2,\mathbb{R})$.

Proposition 3.1.5. *Let $\tilde{L}_n \in PSL(2,\mathbb{R})$ be a sequence converging to $[[\mathrm{id}]]$ and, for each $n \in \mathbb{N}$ let η_n be an $\tilde{L}_n$-invariant measure on $\mathbb{P}^1$ converging to $\frac{1}{2}(\delta_p + \delta_q)$ for some $p,q \in \mathbb{P}^1$ with $p \neq q$. Then for every n sufficiently large either $\tilde{L}_n$ is hyperbolic or $\tilde{L}_n = [[\mathrm{id}]]$.*

This result follows easily from Proposition 3.1.4: for every $\tilde{L}_n \in PSL(2,\mathbb{R})$ we can take a representative of $\tilde{L}_n$ in $SL(2,\mathbb{R})$ with positive trace and apply the aforementioned result to these representatives.

3.2 Proof of Theorem A

Let $f: M \to M$, $A: M \to SL(2,\mathbb{R})$ and μ be given as in Theorem A and suppose there exists a sequence $\{A_k\}_{k\in\mathbb{N}}$ in $H^\alpha(M)$ with $\lambda^u(A_k,\mu) = \lambda^s(A_k,\mu) = 0$ for every $k \in \mathbb{N}$ and such that A_k converges to A in $H^\alpha(M)$.

For each $k \in \mathbb{N}$, let m_k be an ergodic F_{A_k}-invariant probability measure on $M \times \mathbb{P}^1$ projecting on μ where F_{A_k} is defined similarly to F_A. The set $\mathcal{M}(\mu)$ of probability measures on $M \times \mathbb{P}^1$ that project down to μ is sequentially compact (see [24, Lemma 6.4]). Hence, passing to a subsequence if necessary, we may assume that the sequence $\{m_k\}_k$ converges in the weak* topology to some measure m which is, as one can easily check, F_A-invariant and projects on μ. In order to prove Theorem A we are going to analyze these families of measures and its respective disintegrations.

3.2.1 Continuity and convergence of conditional measures

Recall that $\mathcal{W}_L^s = \{(x,y) \in M \times M; x \sim_L^s y\}$ is compact and, since $(x,y,A) \to H_{xy}^{s,A}$ is continuous, given a sequence $\{A_k\}_{k\in\mathbb{N}}$ converging to A in $H^\alpha(M)$,

$$\{\mathcal{W}_L^s \ni (x,y) \to H_{xy}^{s,A_k}\}_{k\in\mathbb{N}}$$

is equi-continuous for k sufficiently large. Then, using this, Proposition 3.1.1 and its proof in [3] we have:

Corollary 3.2.1. *For every k sufficiently large there exists an su-invariant disintegration $\{m_x^k : x \in M\}$ of m_k with respect to the partition $\{\{x\} \times \mathbb{P}^1 : x \in M\}$ and μ such that*

$$\{M \ni x \to m_x^k\}_{k\gg 0} \text{ is equi-continuous.}$$

As an application of this corollary we get that

Proposition 3.2.2. *The measure m is su-invariant and admits a continuous disintegration $\{m_x\}_{x\in M}$ with respect to $\{\{x\}\times\mathbb{P}^1\}_{x\in M}$ and μ so that m_x^k converges uniformly on M to m_x.*

In order to prove the previous proposition we need the following auxiliary result.

Lemma 3.2.3. *Let X and Y be compact metric spaces, μ a Borel probability measure on X and $\{\nu_k\}_{k\in\mathbb{N}}$ be a sequence of probability measures on $X \times Y$ projecting on μ and converging in the weak* topology to some measure ν. Then for every measurable function $\rho : X \to \mathbb{R}$ in $L^1(\mu)$ and every continuous function $\varphi : Y \to \mathbb{R}$,*

$$\lim_{k\to\infty} \int \rho \times \varphi d\nu_k = \int \rho \times \varphi d\nu.$$

Proof. Let φ be a continuous real valued function on Y. It is well-known that the continuous functions are dense in $L^1(\mu)$ (see for instance [21, Appendix A.5]). Thus, given $\varepsilon > 0$, we can take $\hat{\rho} : X \to \mathbb{R}$ a continuous function so that

$$\int_X |\hat{\rho} - \rho| d\mu < \frac{\varepsilon}{2 \sup \varphi}.$$

By the weak convergence, we can take $k_0 \in \mathbb{N}$ such that for every $k > k_0$,

$$\left| \int \hat{\rho} \times \varphi d\nu_k - \int \hat{\rho} \times \varphi d\nu \right| < \frac{\varepsilon}{2}.$$

Then, for $k > k_0$,

$$\left| \int \rho \times \varphi d\nu_k - \int \rho \times \varphi d\nu \right| < \sup \varphi \int_X |\hat{\rho} - \rho| d\mu + \left| \int \hat{\rho} \times \varphi d\nu_k - \int \hat{\rho} \times \varphi d\nu \right| < \varepsilon.$$

□

We now proceed to the proof of Proposition 3.2.2:

Proof of Proposition 3.2.2. For each $k \in \mathbb{N}$, let $\{m_x^k\}_{x\in M}$ be the disintegration of m_k given by Corollary 3.2.1. We start observing that by the equicontinuity of the disintegrations and Arzelà-Ascoli's theorem, for every continuous function $\varphi : \mathbb{P}^1 \to \mathbb{R}$, there exists a subsequence of $\left\{\int_{\mathbb{P}^1} \varphi dm_x^k\right\}_k$ such that

$$\int_{\mathbb{P}^1} \varphi dm_x^{k_j} \to I_x(\varphi)$$

uniformly on M. Moreover, fixing φ, the uniform convergence implies that $I_x(\varphi)$ is continuous in M.

Taking a dense subset $\{\varphi_j\}_{j\in\mathbb{N}}$ of the space $C^0(\mathbb{P}^1)$ of continuous functions and using a diagonal argument, passing to a subsequence if necessary, we can suppose that

$$\int_{\mathbb{P}^1} \varphi dm_x^k \to I_x(\varphi)$$

for every $\varphi \in C^0(\mathbb{P}^1)$. It is easy to see that I_x defines a positive linear functional on $C^0(\mathbb{P}^1)$. Consequently, by Riesz-Markov's theorem, for every $x \in M$ there exists a measure $\hat{m}_x$ on $\mathbb{P}^1$ such that $I_x(\varphi) = \int \varphi d\hat{m}_x$.

On the other hand, letting $\{m_x\}_{x\in M}$ be a disintegration of m with respect to $\{\{x\} \times \mathbb{P}^1\}_{x\in M}$ and μ and invoking Lemma 3.2.3 it follows that for every continuous function $\varphi : \mathbb{P}^1 \to \mathbb{R}$ and any μ-positive measure subset $D \subset M$,

$$\int_D \int_{\mathbb{P}^1} \varphi dm_x^k d\mu = \int_{M\times\mathbb{P}^1} \chi_D \times \varphi dm_k \to \int_{M\times\mathbb{P}^1} \chi_D \times \varphi dm = \int_D \int_{\mathbb{P}^1} \varphi dm_x d\mu.$$

Consequently, $m_x = \hat{m}_x$ for μ almost every $x \in M$. Thus, extending $m_x = \hat{m}_x$ for every $x \in M$ we get a continuous disintegration of m such that $m_x^k \to m_x$ uniformly on $x \in M$. In particular, by the equicontinuity of the holonomies and the su-invariance of m_k for every k it follows that m is also su-invariant as claimed. □

From now on we work exclusively with the disintegrations $\{m_x^k\}_{x\in M}$ and $\{m_x\}_{x\in M}$ of m_k and m, respectively, given by Corollary 3.2.1 and the previous proposition.

Recall we are assuming $\lambda^u(A,\mu) > 0 > \lambda^s(A,\mu)$ hence, by the Oseledets' theorem there exist a decomposition $\mathbb{R}^2 = E_x^{s,A} \oplus E_x^{u,A}$. Moreover, we have the following result, for its proof we refer the reader to [4, Proposition 3.1]:

Lemma 3.2.4. *If $\lambda^u(A,\mu) > 0 > \lambda^s(A,\mu)$, every F_A-invariant probability measure m that projects down to μ may be written as a convex combination $am^s + bm^u$ with $a, b \geq 0$ and $a + b = 1$, where m^* is an F_A-invariant measure projecting on μ such that its disintegration $\{m^*_x\}_{x\in M}$ with respect to μ satisfies $m^*_x(E^*_x) = 1$ for $* \in \{s, u\}$.*

It follows from the lemma above that for any F_A-invariant measure m, its conditional measures are of the form $m_x = a\delta_{E^{u,A}_x} + b\delta_{E^{s,A}_x}$ for some $a, b \in [0,1]$ such that $a + b = 1$ where here and in what follows we abuse notation and identify a 1-dimensional linear space E with its class $[E]$ in $\mathbb{P}^1$. Furthermore, Avila, Santamaria, Viana in [3, Theorem D] established that under the hypothesis of Theorem A every su-invariant section of a continuous fiber bundle $\pi : E \to M$ is continuous in M. Thus we have:

Lemma 3.2.5. *There exist continuous and su-invariant functions which coincide with $x \to E^{s,A}_x, E^{u,A}_x$ for μ-almost every point. By su-invariance we mean that for every (admissible) choice of $x, y, z \in M$, $H^{s,A}_{xy}E^*_x = E^*_y$ and $H^{u,A}_{xz}E^*_x = E^*_z$ for $* \in \{s, u\}$.*

Proof. Recall m_k is a F_{A_k}-invariant measure such that $m_k \to m$. Since $\lambda^u(A_k,\mu) = 0$ for every $k \in \mathbb{N}$ we get that $\int \Phi_{A_k} dm_k = 0$ where $\Phi_{A_k} : M \times \mathbb{P}^1 \to \mathbb{R}$ is given by $\Phi_{A_k}(x, v) = \log \frac{\|A_k(x)v\|}{\|v\|}$. On the other hand,

$$\int \Phi_{A_k} dm_k \to \int \Phi_A dm.$$

Thus, $\int \Phi_A dm = 0$ which implies that

$$a\lambda_+(A,\mu) + b\lambda_-(A,\mu) = \int \Phi_A dm = 0.$$

Furthermore, since $A(x) \in SL(2,\mathbb{R})$ for all $x \in M$ we have $\lambda_+(A,\mu) = -\lambda_-(A,\mu)$. Therefore, $a = b = 1/2$. Now, by Proposition 3.2.2 we know that $\{m_x\}_x$ is su-invariant. Consequently, since $E^{u,A}_x$ is u-invariant and $E^{s,A}_x$ is s-invariant, it follows

$$\delta_{E^{u,A}_x} = \frac{1}{a}(m_x - b\delta_{E^{s,A}_x})$$

is also s-invariant. Analogously, $E^{s,A}_x$ is u-invariant. In particular, $E^{u,A}_x$ and $E^{s,A}_x$ are su-invariant. Continuity follows easily from [3, Theorem D] as mention above. □

From now on we think of $E^{s,A}_x$ and $E^{u,A}_x$ as continuous functions defined for every $x \in M$.

3.2.2 Excluding the atomic case with a bounded number of atoms

In this subsection we prove that $m^k_{x_k}$ can not have a bounded number of atoms (with bound independent of k) for infinitely many values of $k \in \mathbb{N}$ and any $x_k \in M$. The general case will be reduced to this one. In order to do so, we need the following lemma.

Lemma 3.2.6. *If m^k_y has an atom for some $y \in M$, then there exists $j = j(k) \in \mathbb{N}$ such that for every $x \in M$, there exist $v^1_x, \dots v^j_x \in \mathbb{P}^1$ so that*

$$m^k_x = \frac{1}{j} \sum_{i=1}^{j} \delta_{v^i_x}.$$

Proof. Let $v_y \in \mathbb{P}^1$ be such that $m^k_y(v_y) = \beta > 0$ and for every $x \in M$, let γ_x be an su-path joining y and x. Taking $w_x = H^{A_k}_{\gamma_x} v_y$, by the su-invariance of the disintegration $\{m^k_x\}_k$ it follows that $m^k_x(w_x) = \beta$ for every $x \in M$. Thus, considering

$$L = \{(x, v_x) \in M \times \mathbb{P}^1;\ m^k_x(v_x) = \beta\}$$

we get that it is F_{A_k}-invariant. Indeed,

$$m^k_{f(x)}(\mathbb{P}A(x)v_x) = (\mathbb{P}A(x))_* m^k_x(\mathbb{P}A(x)v_x) = m^k_x(v_x) = \beta$$

Consequently, since m_k is ergodic and

$$m_k(L) = \int m^k_x \left(L \cap \left(\{x\} \times \mathbb{P}^1\right)\right) d\mu \geq \beta > 0.$$

it follows that $m_k(L) = 1$. In particular, $m^k_x\left(L \cap (\{x\} \times \mathbb{P}^1)\right) = 1$ for μ-almost every $x \in M$. Otherwise, if the set $D \subset M$ where $m^k_x\left(L \cap (\{x\} \times \mathbb{P}^1)\right) < 1$ has positive measure, then

$$\begin{aligned}
m^k(L) &= \int_M m^k_x \left(L \cap \left(\{x\} \times \mathbb{P}^1\right)\right) d\mu(x) \\
&= \int_D m^k_x \left(L \cap \left(\{x\} \times \mathbb{P}^1\right)\right) d\mu(x) + \int_{D^c} m^k_x \left(L \cap \left(\{x\} \times \mathbb{P}^1\right)\right) d\mu(x) \\
&< \int_D d\mu(x) + \int_{D^c} d\mu(x) = 1.
\end{aligned}$$

By the definition of L, this implies that

$$m^k_x = \frac{1}{j} \sum_{i=1}^{j} \delta_{v^i_x},$$

where $\frac{1}{j} = \beta$ (in particular, j does not depend on x).

This is true for every x in a full measure set $\hat{M} \subset M$. To prove that this claim holds true *for every* $x \in M$, given $x \in M$ take some su-path γ_x from a point $z \in \hat{M}$ to x. By su-invariance, if we define $\omega_x^i = H_{\gamma_x}^A v_z^i$ we have

$$m_x^k = \left(H_{\gamma_x}^A\right)_* m_z^k = \frac{1}{j}\sum_{i=1}^{j} \delta_{\omega_x^i}.$$

□

We now proceed to prove that for infinitely many k non conditional measure of m_k can have a bounded number of atoms with bound independent of k. The proof is going to be by contradiction. Assume there exist a subsequence k's such that there exist $x \in M$ such that m_x^k has a bounded number of atoms.

So, passing to a subsequence and using the previous lemma suppose m_x^k has $j(k)$ atoms and that the sequence $\{j(k)\}_k$ is bounded. Restricting again to a subsequence, if necessary, we may assume that $j(k)$ is constant equal to some $j \in \mathbb{N}$. In particular, since

$$m_x = \frac{1}{2}\delta_{E_x^{s,A}} + \frac{1}{2}\delta_{E_x^{u,A}},$$

for k sufficiently large m_x^k has an even number of atoms. Thus, writing

$$m_x^k = \frac{1}{j}\sum_{i=1}^{j} \delta_{v_k^i(x)}$$

and reordering if necessary we may suppose that $v_k^i(x) \to E_x^{u,A}$ for $i \leq \frac{j}{2}$ and $v_k^\ell(x) \to E_x^{s,A}$ for $\ell > \frac{j}{2}$. Moreover, by Proposition 3.2.2, such convergence is uniform.

Observe now that for each k there exists some $x_k \in M$ such that $A_k(x_k)v_k^{i_k}(x_k) = v_k^{j_k}(f(x_k))$ for some $i_k \leq \frac{j}{2}$ and $j_k > \frac{j}{2}$. Otherwise, the set

$$L = \bigcup_{x \in M} \{x\} \times \{v_k^1(x), \dots v_k^{\frac{j}{2}}(x)\}$$

would be F_{A_k}-invariant with measure

$$m_k(L) = \int m_x^k(\{v_k^1(x), \dots v_k^{\frac{j}{2}}(x)\})d\mu = \frac{1}{2},$$

contradicting the ergodicity. Thus, restricting to a subsequence, if necessary, we may assume without loss of generality that $v_k^{i_k}(x_k) = v_k^1(x_k)$ and $v_k^{j_k}(x_k) = v_k^j(x_k)$ for every $k \in \mathbb{N}$ and that $x_k \to x$. In particular,

$$A(x)E_x^{u,A} = \lim_{k\to\infty} A_k(x_k)v_k^1(x_k) = \lim_{k\to\infty} v_k^j(f(x_k)) = E_{f(x)}^{s,A},$$

a contradiction.

Summarizing, we can not have a subsequence $\{k_i\}_i$ so that the sequence $\{j(k_i)\}_i$ is bounded where $j(k)$ stands for the number of atoms of m_x^k (which is independent of $x \in M$).

3.2.3 Conclusion of the proof

Now we consider the general case. The idea of the proof is to use Proposition 3.1.5 to reduce the proof to the case presented in Section 3.2.2,

Given $x \in M$ let γ be a non-trivial su-loop at x. In particular, from Lemma 3.2.5 it follows that

$$H_\gamma^A E_x^{*,A} = E_x^{*,A}$$

for $* \in \{s, u\}$. Consequently, either H_γ^A is hyperbolic or $H_\gamma^A = \pm\,\mathrm{id}$.

If there exist x and γ such that H_γ^A is hyperbolic then, by the convergence of $H_\gamma^{A_k}$ to H_γ^A, it follows that $H_\gamma^{A_k}$ is also hyperbolic for every $k \gg 0$. Thus, since $\left(H_\gamma^{A_k}\right)_* m_x^k = m_x^k$, it follows that m_x^k is atomic and has at most two atoms for every $k \gg 0$ but from Section 3.2.2 we know this is not possible. So, we get that $H_\gamma^A = \pm\,\mathrm{id}$ *for every* su-loop at x and *every* $x \in M$ and therefore $H_\gamma^{\tilde{A}} = [[\mathrm{id}]]$ *for every* su-loop at x and *every* $x \in M$ where $\tilde{A}$ is defined in 3.1.3.

From Proposition 3.1.5 we get that either there exists a non-trivial su-loop γ at some point $x \in M$ and a sequence $\{k_j\}_j$ going to infinite as $j \to +\infty$ so that $H_\gamma^{\tilde{A}_{k_j}}$ is hyperbolic for every j and thus $H_\gamma^{A_{k_j}}$ is also hyperbolic for every j, or $H_\gamma^{\tilde{A}_k} = [[\mathrm{id}]]$ for every su-loop γ and every $k > k_\gamma$ for some $k_\gamma \in \mathbb{N}$.

Arguing as we did above we conclude that the first case can not happen. So, all we have to analyze is the case when $H_\gamma^{\tilde{A}_k} = [[\mathrm{id}]]$ for every su-loop γ and every $k > k_\gamma$ for some $k_\gamma \in \mathbb{N}$.

If there exists $k_0 \in \mathbb{N}$ so that $k_\gamma \leq k_0$ for every su-loop γ then $H_\gamma^{\tilde{A}_k} = [[id]]$ for all $k > k_0$ and for all γ. Making the change of coordinates given in 3.1 for every $k > k_0$ (recall Section 3.1.3) we get the that $L(\tilde{A}_k, \mu)$ is equal to the logarithm of the norm of the greatest eigenvalue of any representative of $\hat{\tilde{A}}_k(x)$, where $\hat{\tilde{A}}_k(x)$ is a constant element of $PSL(2, \mathbb{R})$, and $\hat{\tilde{A}}_k(x) \to \hat{\tilde{A}}(x)$. In particular,

$$\lambda^u(A_k, \mu) = L(\tilde{A}_k, \mu) \xrightarrow{k \to +\infty} L(\tilde{A}, \mu) = \lambda^u(A, \mu)$$

which is a contradiction.

Recall that in order to perform the change of coordinates in 3.1 it is enough to assume that $H_\gamma^{\tilde{A}_k} = [[\mathrm{id}]]$ for every (K', L')-loop γ for some $K', L' > 0$. To conclude the proof of Theorem A, in view of the previous argument, we only have to show that we can not have k_γ arbitrarily large for (K', L')-loops.

Let k_γ be minimum for its defining property, that is, $H_\gamma^{\tilde{A}_k} = [[\mathrm{id}]]$ for every $k > k_\gamma$ and $H_\gamma^{\tilde{A}_{k_\gamma}} \neq [[\mathrm{id}]]$. Suppose that for each $j \in \mathbb{N}$ there exist $x_j \in M$ and a (K', L')-loop γ_j at x_j so that $k_{\gamma_j} \xrightarrow{j\to+\infty} +\infty$. Passing to a subsequence we may assume $x_j \xrightarrow{j\to+\infty} x$ and $\gamma_j \xrightarrow{j\to+\infty} \gamma$ where γ is an su-loop at x. This can be done because each γ_j has at most K' legs and each of them with length at most L'. In particular, if γ_j is defined by the sequence $x_j = z_0^j, z_1^j, \ldots, z_{n_j}^j = x_j$ then $n_j \leq K'$ for every j. Thus, passing to a subsequence we may assume $n_j = n \leq K'$ for every $j \in \mathbb{N}$ and $z_i^j \xrightarrow{j\to+\infty} x_i$ for every $i = 1, \ldots, n$, consequently γ is the su-loop defined by the sequence $x = x_0, x_1, \ldots, x_n = x$.

Now, since $H_\gamma^{\tilde{A}} = [[\mathrm{id}]]$, $H_{\gamma_j}^{\tilde{A}_{k_{\gamma_j}}} \xrightarrow{j\to+\infty} H_\gamma^{\tilde{A}}$ and $H_{\gamma_j}^{\tilde{A}_{k_{\gamma_j}}} \neq [[\mathrm{id}]]$ it follows from Proposition 3.1.5 (recall Proposition 3.2.2) that $H_{\gamma_j}^{\tilde{A}_{k_{\gamma_j}}}$ is hyperbolic for every $j \gg 0$ and thus $H_{\gamma_j}^{A_{k_{\gamma_j}}}$ is also hyperbolic for every $j \gg 0$. Consequently, $m_x^{k_{\gamma_j}}$ is atomic and has at most two atoms for every $x \in M$ and every $j \in \mathbb{N}$ which again from Section 3.2.2 we know is not possible concluding the proof of Theorem A.

Remark 3.2.7. *We observe that Theorem A can also be proved using the techniques of couplings and energy developed in [5]. We chose to present the previous proof because it is shorter and also different. It is also worth noticing that a similar result was obtained by Liang, Marín and Yang [20, Theorem 6.1] for the derivative cocycle under the additional assumption that f has a pinching hyperbolic periodic point. In our context, such a hypothesis would immediately imply that all the conditional measures m_x^k are atomic with at most two atoms for every $k \gg 0$. In particular, Theorem A would follow from the results of Section 3.2.2.*

3.3 Examples

At this section we present two examples of fiber-bunched cocycles with nonvanishing Lyapunov exponents over a partially hyperbolic map which are accumulated by cocycles with zero Lyapunov exponents. The construction of this examples is based on an example constructed by Wang and You in [26, Theorem 2]. Let us present this result.

Let X be a C^r, $r \geq 1$, compact manifold, $T : X \to X$ be ergodic with a normalized invariant measure ν and $A(x)$ be a $SL(2, \mathbb{R})$-valued function on X. We say that the dynamical system $(x, v) \mapsto (T(x), A(x)v)$ in $X \times \mathbb{R}^2$ is a *quasi periodic cocycle* if the base of the system is a rotation on the torus. That is, if $X = \mathbb{T}^m = \mathbb{R}^m \setminus \mathbb{Z}^m$ and

$$T = T_\omega : x \mapsto x + \omega.$$

Let ω be an irrational number. We call a rational number p/q a *best approximation* of ω if every other rational fraction with the same or smaller denominator differs from ω by a greater amount. In other words if the inequalities $0 < s \leq q$, and $p/q \neq r/s$ imply that

$$\left|\omega - \frac{p}{q}\right| < \left|\omega - \frac{r}{s}\right|.$$

Irrational numbers have an unique continued fraction expansion:

$$\omega = [a_0 : a_1, a_2 ...] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

where $a_0 \in \mathbb{Z}$ and $a_n \in \mathbb{N}$ for $n \geq 1$. The continued fraction expansion of an irrational number is infinite, whereas rational numbers have finite but not unique continued fraction expansions. For the proof of the properties of the continued fraction expansion that we mention here we refer to [18].

Given an irrational number $\omega = [a_0 : a_1, a_2, ...]$ the n-th convergent is the rational number $p_n/q_n = [a_0 : a_1, ..., a_n]$. The sequence $(p_n/q_n)_n$ converges to ω and this sequence is the best approximation to ω: for any other rational number a/b such that $b \leq q_n$ we have

$$\left|\omega - \frac{p_n}{q_n}\right| < \left|\omega - \frac{a}{b}\right|.$$

Moreover, we say ω is of *bounded type* if there exist $M > 0$ such that $a_n \leq M$ for all n.

With this definitions we are able to state Wang and You's result.

Theorem 3.3.1. *[26, Theorem 1] Consider quasi-periodic $SL(2,\mathbb{R})$ cocycles over S^1 with ω being a fixed irrational number of bounded-type. For any $0 \leq l \leq \infty$, there exists a cocycle $A^l \in C^l(S^1, SL(2,\mathbb{R}))$ with arbitrarily large Lyapunov exponent and a sequence of cocycles $A_k \in C^l(S^1, SL(2,\mathbb{R}))$ with zero Lyapunov exponent such that $A_k \to A^l$ in the C^l topology. As a consequence, the set of $SL(2,\mathbb{R})$-cocycles with positive Lyapunov exponent is not C^l open.*

3.3.1 Proof of Theorem B

Let ω be an irrational number of bounded type and $f_0 : \mathcal{S}^1 \to \mathcal{S}^1$ be given by $f_0(t) = t + 2\pi\omega$ where $\mathcal{S}^1$ is the unit circle. Let $A_0 : \mathcal{S}^1 \to \mathrm{SL}(2,\mathbb{R})$ be the cocycle A^r given by Theorem 3.3.1. And $\{A_k\}_k$ be a sequence in $C^r(\mathcal{S}^1, \mathrm{SL}(2,\mathbb{R}))$ converging to A_0 in the C^r topology, so that $\lambda^u(A_k, \nu) = 0$ for every $k \in \mathbb{N}$ where ν denotes the Lebesgue measure on $\mathcal{S}^1$ (also given by Theorem 3.3.1) .

Now, given $f_1 : N \to N$, a volume-preserving Anosov diffeomorphism of a compact manifold N, let us consider the map $f : M := \mathcal{S}^1 \times N \to M$ given by

$$f(t,x) = (f_0(t), f_1(x))$$

and let $\hat{A} : M \to SL(2,\mathbb{R})$ be given by

$$\hat{A}(t,x) = A_0(t).$$

Thus, defining $\hat{A}_k(t,x) = A_k(t)$ and denoting by μ the Lebesgue measure on M we have that the limit of $\hat{A}_k$ is $\hat{A}$. Moreover,

$$\lambda^u(\hat{A}_k, \mu) = \lambda^u(A_k, \nu) = 0$$

for every $k \in \mathbb{N}$ and

$$\lambda^u(\hat{A}, \mu) = \lambda^u(A_0, \nu) > 0.$$

Consequently, since f is a volume-preserving partially hyperbolic and center-bunched diffeomorphism and f_1 may be chosen so that $(\hat{A}, f)$ is fiber-bunched, we complete the proof of Theorem B.

3.3.2 Random product cocycles

We now present another construction showing that given any real number $\lambda > 0$, we have a fiber-bunched cocycle A over a partially hyperbolic and center-bunched map f so that $\lambda^u(A, \mu) = \lambda$ which can be approximated by cocycles with zero Lyapunov exponents. We start with a general construction.

Let $\Sigma = \{1, \dots, k\}^{\mathbb{Z}}$ be the space of bilateral sequences with k symbols and $\sigma : \Sigma \to \Sigma$ be the left shift map. Given maps $f_j : K \to K$ and $A_j : K \to SL(2,\mathbb{R})$ for $j = 1, \dots, k$ where K is a compact manifold, let us consider $f : \Sigma \times K \to \Sigma \times K$ and $A : \Sigma \times K \to SL(2,\mathbb{R})$ given, respectively, by

$$f(x,t) = (\sigma(x), f_{x_0}(t)) \quad \text{and} \quad A(x,t) = A_{x_0}(t).$$

The *random product of the cocycles* $\{(A_j, f_j)\}_{j=1}^{k}$ is then defined as the cocycle over f which is generated by A. Observe that this definition generalizes the notion of random products of matrices explaining our terminology. Indeed, taking K as being a single point we recover the aforementioned notion.

Differently from the case of random products of matrices where one have continuity of Lyapunov exponents (see [5],[9], [24]), in the setting of random products of cocycles Lyapunov exponents can be very 'wild'. This is what we exploit to construct our next example.

Let $f_0 : \mathcal{S}^1 \to \mathcal{S}^1$ and ν be as in the previous example and let $A_0 \in C^r(\mathcal{S}^1, SL(2,\mathbb{R}))$ be given by Theorem 3.3.1 so that $\lambda^u(A_0,\nu) > \lambda$.

Taking $f_1 : \mathcal{S}^1 \to \mathcal{S}^1$ to be $f_1(t) = t$ and $A_1 : \mathcal{S}^1 \to SL(2,\mathbb{R})$ given by $A_1(t) = \mathrm{id}$, let (A, f) be the random product of the cocycles (A_0, f_0) and (A_1, f_1) as defined above. Thus, letting η be the Bernoulli measure on Σ defined by the probability vector (p_0, p_1) where p_0 is so that $p_0\lambda^u(A_0,\nu) = \lambda$ and considering $\mu = \eta \times \nu$, we are going to prove that the cocycle generated by A over f has positive Lyapunov exponents and is accumulated by cocycles with zero Lyapunov exponents.

Indeed, let $\{A_{0,k}\}_k$ be a sequence in $C^r(\mathcal{S}^1, SL(2,\mathbb{R}))$ converging to A_0 for which the cocycle $(A_{0,k}, f_0)$ satisfies $\lambda^u(A_{0,k},\nu) = 0$ for every $k \in \mathbb{N}$ whose existence is guaranteed by our choice of A_0 and Theorem 3.3.1. Also let $\{A_{1,k}\}_k$ be the sequence such that $A_{1,k} = \mathrm{id}$ for every $k \in \mathbb{N}$ and (A_k, f) be the random product of $(A_{0,k}, f_0)$ and $(A_{1,k}, f_1)$. It is easily to see that $A_k \xrightarrow{k\to\infty} A$.

Now, for μ-almost every $(x,t) \in \Sigma \times \mathcal{S}^1$,

$$\lambda^u(A_k,\mu,x,t) = \lim_{n\to\infty} \frac{1}{n} \log \|A_k^n(x,t)\|.$$

Thus, observing that $A_k^n(x,t) = A_{0,k}^{\tau_n(x)}(t)$ where

$$\tau_n(x) = \#\left\{1 \le j \le n;\ \sigma^j(x)_0 = 0\right\},$$

it follows that

$$\lambda^u(A_k,\mu,x,t) = \lim_{n\to\infty} \frac{\tau_n(x)}{n}\frac{1}{\tau_n(x)} \log \left\|A_{0,k}^{\tau_n(x)}(t)\right\| = p_0\lambda^u(A_{0,k},\nu).$$

In particular, $\lambda^u(A_k,\mu,x,t)$ is constant equal to $\lambda^u(A_k,\mu)$ for μ-almost every $(x,t) \in \Sigma \times \mathcal{S}^1$. Analogously, $\lambda^u(A,\mu) = p_0\lambda^u(A_0,\nu)$. Consequently,

$$\lambda^u(A_k,\mu) = 0 \text{ for every } k \in \mathbb{N} \text{ and } \lambda_u(A,\mu) = \lambda > 0$$

as claimed. Observe that despite the fact of not being smooth, the map f is partially hyperbolic in the sense of the expansion and contraction properties when Σ is endowed with the usual metric. Moreover, it is center-bunched and the cocycle A is fiber-bunched.

Probability distributions with non compact support

In this chapter we present our results regarding the continuity of Lyapunov exponents when the base measure has non compact support. We first analyze the semicontiuity relative to the weak* topology and to the Wasserstein topology in Section 4.1. Finally, in Section 4.2, we prove that the Wasserstein topology is not enough to guarantee the continuity of the Lyapunov exponents.

4.1 Semicontinuity

Let us begin by giving an example that shows that if we only assume convergence in the weak* topology, the Lyapunov exponents are not necessary semicontinuous. On the other hand, in Section 4.1.2 we use the convergence of the first moments provided by the Wassertein topology, to prove the semicontinuity relative to this topology.

4.1.1 Semicontinuity counterexample with weak* topology

At this section we proof Theorem 2.4.3. Thus, we construct a sequence of measures q_k converging to q in the weak* topology such that $\lambda_+(q_k) \geq 1$ while $\lambda_+(q) = 0$.

We begin by defining the function $\alpha : \mathbb{N} \to SL(2, \mathbb{R})$ by

$$\alpha(2k-1) = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k^{-1} \end{pmatrix}$$

$$\alpha(2k) = \begin{pmatrix} \sigma_k^{-1} & 0 \\ 0 & \sigma_k \end{pmatrix}$$

where $(\sigma_k)_k$ is an increasing sequence such that $\sigma_1 > 1$ and $\sigma_k \to +\infty$.

Let $\mu = q^{\mathbb{Z}}$ be a measure in M where q is the measure on $SL(2, \mathbb{R})$ given by

$$q = \sum_{k \in \mathbb{N}} p_k \delta_{\alpha(k)},$$

with $\sum p_k = 1$, $0 < p_k < 1$ for all $k \in \mathbb{N}$.

The key idea to construct this example is to find p_k and σ_k such that $\log \|A\| \in L^1(\mu)$ and satisfying the hypothesis above. Consider $0 < r < 1/2 < s < 1$, and $l = s/r > 1$. Let us take $\sigma_k = e^{l^k}$ for all k, which is an increasing sequence provided that $l > 1$.

For $k \geq 2$ take $p_{2k-1} = p_{2k} = r^k$. Since $0 < r < 1/2$ it is easy to see that

$$\sum_{k\geq 3} p_k = 2\sum_{k\geq 2} p_{2k} = 2\sum_{k\geq 2} r^k = 2\frac{r^2}{1-r} < 1$$

We have to choose p_1 and p_2 such that $\sum p_k = 1$. Then, it is enough to take

$$p_1 = p_2 = \frac{1}{2}\left(1 - 2\frac{r^2}{1-r}\right).$$

We continue by showing that $\log \|A\| \in L^1(\mu)$. This is an easy computation,

$$\int_M \log \|A\| d\mu = 2p_2 \log \sigma_1 + 2\sum_{k\geq 2} p_{2k} \log \sigma_k = 2p_2 l + 2\sum_{k\geq 2} s^k.$$

Since $0 < s < 1$ this geometric series is convergent. Moreover, since $p_{2k-1} = p_{2k}$ for all k then $\lambda_+(q) = 0$.

What is left is to construct the sequence q_n. Fix $n_0 > 1$ large enough so $\frac{1}{2}\left(1 - 2\frac{r^2}{1-r}\right) > l^{-n}$ for all $n \geq n_0$, and consider $q_n = \sum_k q_k^n \delta_{\alpha(k)}$ where for $n \geq n_0$

$$\begin{aligned} q_{2n}^n &= l^{-n} + r^n, \\ q_2^n &= \frac{1}{2}\left(1 - \frac{r^2}{1-r}\right) - l^{-n} \\ q_k^n &= p_k \text{ other case.} \end{aligned}$$

Thus, since q_k^n converges to p_k when n goes to infinite for all k, it is easy to see that q_n converges in the weak* topology to q.

The proof is completed by showing that $\lambda_+(q_n) \geq 1$ for n large enough. It follows easily since,

$$\lambda_+(q_n) = |q_{2n-1}^n - q_{2n}^n| \log \sigma_n + |q_1^n - q_2^n| \log \sigma_1 = l^{-n} l^n + l^{-n+1},$$

which is equal to $1 + l^{-n+1} \geq 1$ for all $n \geq n_0$.

4.1.2 Semicontinuity relative to the Wasserstein topology

We now consider the Wasserstein topology in $P_1(SL(2,\mathbb{R}))$. The advantage of using this topology is that all probability measures in $P_1(SL(2,\mathbb{R}))$ have finite moment of order 1. Therefore, the Lyapunov exponents always exist. This observation is a direct consequence of the fact that $\log : [1, \infty) \to \mathbb{R}$ is a 1-Lipschitz function and, $\|\alpha\| \geq 1$ for every matrix $\alpha \in SL(2, \mathbb{R})$, because

$$\int \log \|A(x)\| d\mu = \int \log \|\alpha\| dp \leq \int d(\alpha, \mathrm{id}) dp < \infty.$$

Before beginning the proof of Theorem 2.4.4 we need to recall some important results regarding the relationship between Lyapunov exponents and stationary measures.

A probability measure η on $\mathbb{P}^1$ is called a p-**stationary** if

$$\eta(E) = \int \eta(\alpha^{-1} E) dp(\alpha),$$

for every measurable set $E \in \mathbb{P}^1$ and $\alpha^{-1}E = \{[\alpha^{-1}v] : [v] \in E\}$.

Roughly speaking, the following result shows that the set of stationary measures for a measure p is closed for the weak* topology.

Proposition 4.1.1. *Let $(p_k)_k$ be probability measures in $SL(2, \mathbb{R})$ converging to p in the weak* topology. For each k, let η_k be p_k-stationary measures and η_k converges to η in the weak* topology. Then η is a stationary measure for p.*

Furthermore, when f is the shift map, it is well-known that

$$\lambda_+(p) = \max \left\{ \int \Phi dp \times \eta : \eta \ \ p - \text{stationary} \right\}, \tag{4.1}$$

where $\Phi : SL(2, \mathbb{R}) \times \mathbb{P}^1 \to \mathbb{R}$ is given by

$$\Phi(\alpha, [v]) = \log \frac{\|\alpha v\|}{\|v\|}.$$

For more details see for example [24, Proposition 6.7].

Let us begin the proof of Theorem 2.4.4. In order to do this we will prove that $\lambda_+(p)$ is upper semi-continuous. The case of $\lambda_-(p)$ is analogous.

Let $(p_k)_k$ be a sequence in the Wasserstein space $P_1(M)$ converging to p, that is $W(p_k, p) \to 0$. For each $k \in \mathbb{N}$ let η_k a stationary measure that realizes the maximum in (4.1). That is:

$$\lambda_+(p_k) = \int \Phi dp_k d\eta_k.$$

Since $\mathbb{P}^1$ is compact then $\mathcal{M}(\mathbb{P}^1)$ the set of all invariant measures in $\mathbb{P}^1$ is sequentially compact (see [21, Proposition 2.1.6]). Then, passing to a subsequence if necessary, we can suppose η_k converges in the weak* topology to a measure η which, as established by Proposition 4.1.1, is a p-stationary measure.

Let $\epsilon > 0$, we want to prove that there exist a constant $k_0 \in \mathbb{N}$ such that for each $k > k_0$

$$\left| \int \Phi dp_k d\eta_k - \int \Phi dp d\eta \right| < \epsilon.$$

In order to do this we need to consider some properties of the Wasserstein topology. First of all, since the first moment of p is finite, there exist K_1 a compact set of $SL(2,\mathbb{R})$ such that

$$\int_{K_1^c} d(\alpha, \mathrm{id}) dp < \frac{\epsilon}{36}. \tag{4.2}$$

Moreover, by Proposition 2.3.1, since $p_k \xrightarrow{W} p$ there exist $R' > 0$ satisfying

$$\limsup_k \int_{d(\alpha,\mathrm{id})>R'} d(\alpha, \mathrm{id}) dp_k < \frac{\epsilon}{36},$$

then, there exist $k' > 0$ such that for every $k > k'$

$$\int_{d(\alpha,\mathrm{id})>R'} d(\alpha, \mathrm{id}) dp_k < \frac{\epsilon}{36}. \tag{4.3}$$

Take $R > 0$ big enough so $B(\mathrm{id}, R') \cup K_1 \subset B(\mathrm{id}, R)$ and define the compact set $K = \bar{B}(\mathrm{id}, R)$.

Since the function $\log : [1, \infty) \to \mathbb{R}$ is 1-Lipschitz and $\|\alpha\| \geq 1$ for all $\alpha \in SL(2,\mathbb{R})$, then

$$|\Phi(\alpha, [v])| = \left| \log \frac{\|\alpha v\|}{\|v\|} \right| \leq \log \|\alpha\| \leq |\|\alpha\| - \|\,\mathrm{id}\,\|| \leq d(\alpha, \mathrm{id}). \tag{4.4}$$

Our proof starts with the observation that

$$\begin{aligned} &\left| \int \Phi dp_k d\eta_k - \int \Phi dp d\eta \right| \\ &\leq \left| \int_{K\times\mathbb{P}^1} \Phi dp_k d\eta_k - \int_{K\times\mathbb{P}^1} \Phi dp d\eta \right| + \left| \int_{K^c\times\mathbb{P}^1} \Phi dp_k d\eta_k \right| + \left| \int_{K^c\times\mathbb{P}^1} \Phi dp d\eta \right|. \end{aligned}$$

On account of (4.3) it follows that

$$\left| \int_{K^c\times\mathbb{P}^1} \Phi dp_k d\eta_k \right| \leq \int_{K^c} d(\alpha, \mathrm{id}) dp_k < \frac{\epsilon}{3}. \tag{4.5}$$

Furthermore, (4.2) implies that

$$\left| \int_{K^c\times\mathbb{P}^1} \Phi dp d\eta \right| \leq \int_{K^c} d(\alpha, \mathrm{id}) dp < \frac{\epsilon}{3}. \tag{4.6}$$

We now proceed to analyze the integral:

$$\begin{aligned}
&\left|\int_{K\times\mathbb{P}^1}\Phi dp_k d\eta_k-\int_{K\times\mathbb{P}^1}\Phi dp d\eta\right| \\
&\leq\left|\int_{K\times\mathbb{P}^1}\Phi dp_k d\eta_k-\int_{K\times\mathbb{P}^1}\Phi dp_k d\eta\right|+\left|\int_{K\times\mathbb{P}^1}\Phi dp_k d\eta-\int_{K\times\mathbb{P}^1}\Phi dp d\eta\right|.
\end{aligned}$$

Consider $\Phi_K=\Phi|_{K\times\mathbb{P}^1}$ the restriction of Φ to the compact space $K\times\mathbb{P}^1$. Then, Φ_K is uniformly continuous with the product metric. Hence, there exist $\delta=\delta(\epsilon)$ such that for every $[v]\in\mathbb{P}^1$ and every $\alpha,\beta\in K$ satisfying $d(\alpha,\beta)<\delta$ we have

$$|\Phi_K(\alpha,[v])-\Phi_K(\beta,[v])|<\frac{\epsilon}{18}.$$

Moreover, by the compactness of the set K we can find $\alpha_1,...,\alpha_N\in K$ such that $K\subset\cup_{i=1}^N B(\alpha_i,\delta)$. Therefore, the convergence of $(\eta_k)_k$ to η in the weak* topology implies that for each $i=1,...,N$ there exist $k_i>0$ such that for $k>k_i$

$$\left|\int_{\mathbb{P}^1}\Phi_K(\alpha_i,[v])d\eta_k-\int_{\mathbb{P}^1}\Phi_K(\alpha_i,[v])d\eta\right|<\frac{\epsilon}{18}.$$

Take $k''=\max\{k_1,...,k_N\}$. From the above it follows that given $\alpha\in K$ there exist i such that $d(\alpha,\alpha_i)<\delta$ and for every $k>k''$ if $\Phi_i([v])=\Phi_K(\alpha_i,[v])$ then

$$\begin{aligned}
&\left|\int_{\mathbb{P}^1}\Phi_K(\alpha,\cdot)d\eta_k-\int_{\mathbb{P}^1}\Phi_K(\alpha,\cdot)d\eta\right| \\
&\leq\int_{\mathbb{P}^1}|\Phi_K(\alpha,\cdot)-\Phi_i|\,d\eta_k+\left|\int_{\mathbb{P}^1}\Phi_i d\eta_k-\int_{\mathbb{P}^1}\Phi_i d\eta\right| \\
&+\int_{\mathbb{P}^1}|\Phi_i-\Phi_K(\alpha,\cdot)|\,d\eta<\frac{\epsilon}{6}.
\end{aligned}$$

Since this convergence is uniform on α, this implies that

$$\left|\int_{K\times\mathbb{P}^1}\Phi_K(\alpha,[v])d\eta_k dp_k-\int_{K\times\mathbb{P}^1}\Phi_K(\alpha,[v])d\eta dp_k\right|<\frac{\epsilon}{6}, \tag{4.7}$$

for all $k>k''$.

Now, for each $n\in\mathbb{N}$ define $A_n=SL(2,\mathbb{R})\backslash B(\mathrm{id},R+1/n)$ and, consider the Urysohn function $f_n:SL(2,\mathbb{R})\to[0,1]$ given by

$$f_n(\alpha)=\frac{d(\alpha,A_n)}{d(\alpha,A_n)+d(\alpha,K)},$$

which converges pointwise to χ_K, the characteristic function on K. It is easily seen that f_n is continuous for each n, equal to zero in A_n and equal to 1 in K. Therefore, the functions

$$\varphi_n(\alpha)=\int_{\mathbb{P}^1}\Phi(\alpha,[v])d\eta\cdot f_n(\alpha)$$

are continuous. Fix (any) $n \in \mathbb{N}$, then since $|\varphi_n(\alpha)| \leq d(\alpha, \mathrm{id})$ and, by Proposition 2.3.1 (4), there exist $k''' = k'''(n)$ such that for every $k > k'''$

$$\left|\int \varphi_n dp_k - \int \varphi_n dp\right| < \frac{\epsilon}{18}.$$

Moreover, if we denote

$$\varphi(\alpha) = \int_{\mathbb{P}^1} \Phi(\alpha, [v]) d\eta \cdot \chi_K(\alpha).$$

Since $\varphi_n - \varphi = 0$ in K and

$$|\varphi_n - \varphi| \leq \log \|\alpha\| |f_n(\alpha) - \chi_K(\alpha)| \leq 2d(\alpha, \mathrm{id}).$$

By (4.2) and (4.3) we have

$$\int |\varphi_n - \varphi| dp \leq 2 \int_{K^c} d(\alpha, \mathrm{id}) dp < \frac{\epsilon}{18},$$
$$\int |\varphi_n - \varphi| dp_k \leq 2 \int_{K^c} d(\alpha, \mathrm{id}) dp_k < \frac{\epsilon}{18}$$

for each $k > k'$. Thus, if $k > \max\{k', k'''\}$ we get

$$\left|\int_{K\times\mathbb{P}^1} \Phi dp_k d\eta - \int_{K\times\mathbb{P}^1} \Phi dp d\eta\right| < \frac{\epsilon}{6}. \tag{4.8}$$

Finally, taking $k_0 = \max\{k', k'', k'''\}$, by (4.5) - (4.8), we conclude that for every $k > k_0$

$$\left|\int \Phi dp_k d\eta_k - \int \Phi dp d\eta\right| < \epsilon.$$

We just proved that

$$\lambda_+(p_k) = \int \Phi dp_k d\eta_k \to \int \Phi dp d\eta \leq \lambda_+(p),$$

which concludes our proof.

Note that this theorem also implies that the probability measures with zero Lyapunov exponents are points of continuity. This is clear since, in that case,

$$\lambda_+(p) = 0 = \lambda_-(p).$$

Remark 4.1.2. *Theorem 2.4.3 does not contradicts Theorem 2.4.4 since* $p \notin P_1(SL(2, \mathbb{R}))$. *To see this take* $x_0 = \mathrm{id}$*, then*

$$\int d(x, x_0) dp = \sum_{k=0}^{\infty} p_k \|\alpha_k - \mathrm{id}\| = 2 \sum_{k=0}^{\infty} r^k (e^{l^k} - 1)$$

which diverges.

4.2 Examples of discontinuity

In this section we now proceed to the proof of Theorem 2.4.5. The idea of the proof is to construct a measure q with countable support of hyperbolic matrices $(\alpha_k)_k$ and positive Lyapunov exponent, and approximate it with measures q_n whose support is the same as q but changes the matrix α_n by a rotation that exchanges the vertical and horizontal axes. This would allow q_n to have zero Lyapunov exponents.

Moreover, in Sections 4.2.2 and 4.2.3 we present two example of discontinuity points of the Lyapunov exponents in the Wasserstein topology in $SL(2,\mathbb{R})^5$ and $GL(2,\mathbb{R})^2$ respectively. The advantage of having more coordinates is that we can exchange the axes by small rotations guarantying the closeness of the supports.

4.2.1 Proof of Theorem D

Consider the matrix valued function α defined by the hyperbolic matrices

$$\alpha(k) = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix}$$

Take $m \in \mathbb{N}$ the smallest natural number bigger than 1 such that $\sum_{n\geq m} e^{-\sqrt{n}} < 1$, which exist since $\sum_k e^{-\sqrt{k}}$ is convergent, and define

$$\begin{aligned} p_k &= e^{-\sqrt{k}}, \text{ if } k \geq m, \\ p_1 &= 1 - \sum_{n\geq m} e^{-\sqrt{n}}, \\ p_k &= 0, \text{ otherwise.} \end{aligned}$$

It is obvious from the definition that $\sum_k p_k = 1$. Hence, we define the probability measure $q = \sum p_k \delta_{\alpha(k)}$. We need to see that $q \in P_1(SL(2,\mathbb{R}))$, in order to do so notice that if $x_0 = \mathrm{id}$

$$\int d(x, x_0) dq = \sum_k p_k \|\alpha(k) - \mathrm{id}\| = \sum_k e^{-\sqrt{k}}(k-1)$$

which is convergent by the Cauchy condensation test. Moreover, we have

$$\lambda_+(q) = \sum_{k\geq m} e^{-\sqrt{k}} \log k > 0.$$

Now, consider $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and for each n consider

$$\beta_n(k) = \begin{cases} \alpha(k) & \text{if } k \neq n, \\ B & \text{if } k = n. \end{cases}$$

With this we define the probability measures $q_n = \sum_k p_k \delta_{\beta_n(k)}$. In a similar way as above, we can see that for all n these measures belong to $P_1(SL(2,\mathbb{R}))$. We proceed to show that q_n converges to q in the Wasserstein topology. This follows since

$$W(q_n, q) \leq p_n d(\alpha(n), B) \sim ne^{-\sqrt{n}}$$

which goes to 0 if n goes to ∞.

It remains to proof that $\lambda_+(q_n) = 0$ for every n. This could be seen as a direct consequence of a result presented by Bougerol and Lacroix in [11]. Before enunciate this result let us recall a few definitions. We say that a measure q is *irreducible* if there is no proper subspace of $\mathbb{R}^2$ invariant under all the matrices in the support of q. Moreover q is *strongly irreducible* if there does not exist a finite family of proper linear subspaces of $\mathbb{R}^2$ invariant under the support of q.

Proposition 4.2.1. *[11, Theorem 6.1] Assume that the measure q is irreducible. Then, if $\lambda_+(q) > \lambda_-(q)$ one must have q strongly irreducible.*

Here we present a more synthesized proof based on the ideas used by Bougerol and Lacroix on the proof of the result mention above. We proceed by contradiction. Suppose there exist N such that $\lambda_+(q_N) > 0 > \lambda_-(q_N)$. We will consider the distance in the projective space $\mathbb{P}^1$ given by

$$\delta([v],[w]) := \frac{\|v \wedge w\|}{\|v\|\|w\|} = \sin(\angle(v,w)).$$

Consider the family $\mathcal{F} = \{V, H\}$, where $V = [e_2]$ and $H = [e_1]$ are the vertical and horizontal axis respectively. By the definition of the measure q_N, it is clear that this family is invariant by every matrix in the support of this measure. Moreover, if $x = (x_k)_k \in M$ then we can see that for every m

$$\delta(A^m(x)H, A^m(x)V) \geq \delta(H, V) = 1. \tag{4.9}$$

On the other hand, we have for every unit vectors v and w

$$\|A^m(x)v \wedge A^m(x)w\| \leq \| \wedge^2 A^m(x)\|.$$

It is widely known that, for q_N-almost every $x \in M$

$$\lambda_+(q_N) + \lambda_-(q_N) = \lim_m \frac{1}{m} \log \| \wedge^2 A^m(x)\|.$$

Notice that q_N is irreducible. Therefore, we have

$$\lambda_+(q_N) = \lim_m \frac{1}{m} \log \|A^m(x)w\|,$$

for every unit vector w. For a deeper discussion of the two results mention above we refer the reader to [11, Section III.5].

Thus, we have for every unit vectors v and w

$$\lim_m \frac{1}{m} \log \frac{\| \wedge^2 A^m(x)\|}{\|A^m(x)v\|\|A^m(x)w\|} = \lambda_-(q_N) - \lambda_+(q_N) < 0,$$

and hence

$$\begin{aligned}\lim_m \delta(A^m(x)H, A^m(x)V) &\leq \lim_m \frac{\| \wedge^2 A^m(x)\|}{\|A^m(x)e_1\|\|A^m(x)e_2\|} \\ &= \lim_m \exp\left(m \cdot \frac{1}{m} \log \frac{\| \wedge^2 A^m(x)\|}{\|A^m(x)e_1\|\|A^m(x)e_2\|}\right) = 0,\end{aligned}$$

which is a contradiction with (4.9) and, we finish our proof.

Notice that this example shows that the Wasserstein topology is not enough to guarantee continuity of the Lyapunov exponents. The main problem is that the support of the measures q_n move further apart from the support of q. Thus, this suggest that we need to add some hypothesis guaranteeing the "convergence" of the supports. An assumption of this type was made by Bocker, Viana in [9] in order to prove the continuity for measures with compact support.

In the next two sections we are going to describe a construction of points of discontinuity of the Lyapunov exponents as functions of the measure, relative to the Wasserstein topology. However, in each of them the support of the measures are arbitrarily close. These constructions were inspired by the discontinuity example presented by Bocker Viana in [9, Section 7.1].

4.2.2 Discontinuity example in $SL(2,\mathbb{R})^5$

Let us recall that $M = (SL(2,\mathbb{R}))^{\mathbb{Z}}$, $f : M \to M$ is the shift map over M defined by

$$(\alpha_n)_n \mapsto (\alpha_{n+1})_n.$$

And the linear cocycle A is the product of random matrices which is defined by

$$A : M \to SL(2,\mathbb{R}), \quad (\alpha_n)_n \mapsto \alpha_0.$$

Given an invariant measure p in $SL(2,\mathbb{R})$ we can define $\mu = p^{\mathbb{Z}}$ which is an invariant measure in M.

Now consider $X = SL(2,\mathbb{R})^5$ with the product metric

$$d_\infty((\alpha_1, ..., \alpha_5), (\beta_1, ..., \beta_5)) = \max\{d(\alpha_1, \beta_1), ..., d(\alpha_5, \beta_5)\}.$$

Let $N = X^{\mathbb{Z}}$ be the space of sequences over X and $g : N \to N$ the shift map over N. We can identify N with M using the function $\iota : M \to N$ by $\iota((\alpha_n)_n) = (\beta_n)_n$ where

$$\beta_n = (\alpha_{5n}, \alpha_{5n+1}, \alpha_{5n+2}, \alpha_{5n+3}, \alpha_{5n+4}).$$

It is easy to see that ι defines a bijection between N and M. Moreover, we have the following identity

$$g(\iota((\alpha_n)_n)) = f^5((\alpha_n)_n).$$

Also we can consider the linear cocycle induced by A in N, that is the function $B : N \to SL(2,\mathbb{R})$ given by

$$B((\iota((\alpha_n)_n)) = A^5((\alpha_n)_n).$$

So in this context we have the following result.

Theorem 4.2.2. *There exist a measure q and a sequence of measures $(q_n)_n$ on X converging to q in the Warssestein topology, such that*

$$\lambda_+(B, q_n) \nrightarrow \lambda_+(B, q).$$

The main idea of the proof is to construct a measure on N whose Lyapunov exponents are positive and approximate it, in the Warssestein topology, by measures with zero Lyapunov exponents. In order to do that, define the function $\alpha : \mathbb{N} \to SL(2,\mathbb{R})$ as

$$\alpha(2k-1) = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix}$$

$$\alpha(2k) = \begin{pmatrix} k^{-1} & 0 \\ 0 & k \end{pmatrix}$$

As in the example before take $m \in \mathbb{N}$ the smallest natural (odd) number bigger than 3 such that $\sum_{k\geq m} e^{-\sqrt{k}} < 1$, which exist since $\sum_k e^{-\sqrt{k}}$ is convergent, and define

$$\begin{aligned}
&p_{2k} = p_{2k-1} = \frac{1}{2}e^{-\sqrt{k}}, \text{ if } 2k-1 \geq m, \\
&p_3 = 1 - \sum_{n\geq m} e^{-\sqrt{n}}, \\
&p_k = 0, \text{ otherwise.}
\end{aligned}$$

Let $\mu = \tilde{q}^{\mathbb{Z}}$ be a measure in M where $\tilde{q}$ is the measure on $SL(2,\mathbb{R})$ given by

$$\tilde{q} = \sum_{k\in\mathbb{N}} p_k \delta_{\alpha(k)}.$$

Let us consider the space $\Omega = \mathbb{N}^5$ and define the measure on X by

$$q = \sum_{w\in\Omega} p_w \delta_{\alpha(w)},$$

where $\alpha(w) = (\alpha(w_1), \cdots, \alpha(w_5))$ and, $p_w = p_{w_1}\cdots p_{w_5}$ if $w = (w_1, ..., w_5)$.

Now consider the measure $\nu = q^{\mathbb{Z}}$ on N. First, we need to ensure that the measure q belong to $P_1(X)$. This is a direct consequence of the fact that $\sum e^{-\sqrt{n}}(n-1)$ is convergent equal to some positive constant c. Indeed, if $\alpha_0 = (\mathrm{id}, ..., \mathrm{id})$ and the notation $p_1 \cdots \hat{p}_i \cdots p_5$ denotes the product of p_1 through p_5 except p_i then

$$\begin{aligned}
\int d_\infty(\alpha, \alpha_0) dq &= \sum_w p_w d_\infty(\alpha(w), \alpha_0) \\
&< \sum_{i=1}^{5} \sum_{w_j, j\neq i} p_{w_1}\cdots \hat{p}_{w_i}\cdots p_{w_5} \left(\sum_{w_i} p_{w_i} d(\alpha(w_i), \mathrm{id}) \right) \\
&< c\sum_{i=1}^{5} \sum_{w_j, j\neq i} p_{w_1}\cdots \hat{p}_{w_i}\cdots p_{w_5} \\
&= 5c
\end{aligned}$$

which proves our claim. Remember that this also guarantees the existence of $\lambda_\pm(B, q)$ as mention in Section 4.1.2.

It is easy to see that $\nu = \iota_*\mu$. Using this we have

$$\begin{aligned}
\lambda_+(B, q) &= \lim_n \frac{1}{n} \int_M \log \|B^n(i(x))\| d\mu \\
&= \lim_n \frac{1}{n} \int_M \log \|A^{5n}(x)\| d\mu \\
&= 5\lambda_+(A, \tilde{q}) \\
&= 5p_3 \log 2 > 0.
\end{aligned}$$

The task is now to construct the sequence $(q_n)_n$. In order to do this, for each $n \in \mathbb{N}$ consider $w_n = (2n, 2n+2, 2n+1, 2n-1, 2n-1)$ and define

$$\beta(w_n) = (\alpha(2n)R_\epsilon, \alpha(2n+2), \alpha(2n+1)R_\delta, \alpha(2n-1), \alpha(2n-1)R_\epsilon),$$

where $\epsilon = n^{-1}(n+1)^{-1}$, $\delta = \arctan(\epsilon)$ and,

$$R_\epsilon = \begin{pmatrix} 1 & 0 \\ \epsilon & 1 \end{pmatrix}, \quad R_\delta = \begin{pmatrix} \cos(\delta) & -\sin(\delta) \\ \sin(\delta) & \cos(\delta) \end{pmatrix}.$$

We proceed to define the sequence by

$$q_n = \sum_{w \neq w_n} p_w \delta_{\alpha(w)} + p_{w_n} \delta_{\beta(w_n)}.$$

We claim that $W(q_n, q) \to 0$ if n goes to infinite. Our proof starts with the observation that

$$\pi_n = \sum_{w \neq w_n} p_w \delta_{(\alpha(w), \alpha(w))} + p_{w_n} \delta_{(\alpha(w_n), \beta(w_n))}$$

is a coupling of q and q_n. Then,

$$\begin{aligned}
W(q_n, q) &\leq \int d_\infty(u, v) \pi_k(u, v) \\
&= p_{w_n} d(\alpha(w_n), \beta(w_n)) \\
&< \max\{\|\alpha(2n) - \alpha(2n)R_\epsilon\|, \|\alpha(2n-1) - \alpha(2n-1)R_\delta\|, \|\alpha(2n+1) - \alpha(2n+1)R_\epsilon\|\} \\
&\leq \epsilon(n+1) = n^{-1}
\end{aligned}$$

which proofs our claim.

What is left is to show that $\lambda_+(B, q_n) = 0$ for all n. The method of proof follows the same arguments as the Bocker-Viana example, we present it here for completeness. The key idea is to prove the following lemma.

Lemma 4.2.3. *Let $H_x = \mathbb{R}(1,0)$ and $V_x = \mathbb{R}(0,1)$. If $Z_n = [0 : \beta(w_n)]$ then, for all $x \in Z_n$ we have $B(x)H_x = V_{g(x)}$ and $B(x)V_x = H_{g(x)}$*

Proof. Notice that for any $x \in Z_n$

$$B(x) = \begin{pmatrix} 0 & -\epsilon^{-2}\sin(\delta) \\ \epsilon^2 \sin(\delta) + \epsilon\cos(\delta) & 0 \end{pmatrix}.$$

Which completes the proof. ☐

Let $\mu_n = q_n^{\mathbb{Z}}$ and consider η_n its normalized restriction to Z_n. Also consider $g_n : Z_n \to Z_n$ be the first return map. Note that Z_n can be describe by

$$Z_n = \bigsqcup_{b \in \mathcal{B}} [0 : w_n, b, w_n]$$

where the union is over the set $\mathcal{B}$ of all finite words $b = (b_1, ..., b_s)$ with $b_j \neq w_n$ for every $j = 1, ..., s$. Moreover,

$$g_n|_{[0:w_n,b,w_n]} = g^s|_{[0:w_n,b,w_n]}$$

for each $b \in \mathcal{B}$. Therefore, (g_n, η_n) is a Bernoulli shift with alphabet $\mathcal{B}$ and probability determined by $p_b = \eta_k([0 : w_n, b, w_n])$.

Let $\tilde{B} : Z_n \to SL(2, \mathbb{R})$ the cocycle induced by B over g_n. The next lemma establishes that the Lyapunov exponents of $\tilde{B}$ are obtained by multiplying the Lyapunov exponents of B by the average return time. For its proof see [24, Proposition 4.18].

Proposition 4.2.4. *For η_n-almost every $x \in Z_n$ there exist $c(x) \geq 1$ such that the Lyapunov exponents satisfy*

$$\lambda_{\pm}(\tilde{B}, x) = c(x)\lambda_{\pm}(B, x),$$

for η_n-almost every $x \in Z_n$. Moreover, by the ergodicity we have

$$\lambda_{\pm}(\tilde{B}, q_n) = \frac{1}{\eta_n(Z_n)}\lambda_{\pm}(B, q_n).$$

Then it is is sufficient to proof that $\lambda_{\pm}(\tilde{B}, q_n) = 0$ for every n.

Suppose that the Lyapunov exponents of $\tilde{B}$ are different from zero and let $\mathbb{R}^2 = E_x^u \bigoplus E_x^s$ be the Oseledets decomposition. Note that, by Lemma (4.2.3), we have

$$\tilde{B}(x)H_x = V_{g_n(x)} \quad \text{and} \quad \tilde{B}(x)V_x = H_{g_n(x)} \tag{4.10}$$

for η_n-almost every $x \in Z_n$.

Let m_n be the probability measure on $N \times \mathbb{P}^1$ defined by

$$m_n(D) = \frac{1}{2}\eta_n(\{x \in Z_n : H_x \in D\}) + \frac{1}{2}\eta_n(\{x \in Z_n : V_x \in D\}).$$

Thus, m_n is the measure that projects down to η_n and whose disintegration is given by

$$x \mapsto \frac{1}{2}(\delta_{H_x} + \delta_{V_x}).$$

From (4.10) it follows easily that m_n is $\tilde{B}$-invariant. It is a well-known fact that this implies m_n is a linear combination of the measures

$$m_n^u(E) = \eta_n(\{x \in Z_n : (x, E_x^u) \in E\}),$$
$$m_n^s(E) = \eta_n(\{x \in Z_n : (x, E_x^s) \in E\}).$$

The best general reference here is [24, Lemma 5.25].

The only point remaining concerns the ergodicity of m_n. If m_n is ergodic then it must coincide with m_n^s or m_n^u. However, the conditional probability measures of m_n are supported on exactly two points while m_n^s and m_n^u have a single point in their support. This contradiction proves that the Lyapunov exponents of $\tilde{B}$, and hence of B, with respect to q_n vanish for every n. Consequently, the proof is completed by showing that m_n is ergodic.

Suppose by contradiction that m_n is not ergodic. Then, there is an invariant set $D \subset N \times \mathbb{P}^1$ with $0 < m_n(D) < 1$. Let D_0 be the set of points $x \in Z_n$ whose fiber restricted to D, that is $D \cap (\{x\} \times \mathbb{P}^1)$, contains neither H_x nor V_x. This set is (g_n, η_n)-invariant on account of (4.10). Thus, since η_n is ergodic then the η_n-measure of D_0 is either 0 or 1. Since $m_n(D) > 0$ then, by the definition of m_n, we must have $\eta_n(D_0) = 0$. By the same method, since $m_n(D) < 1$, it follows that the η_n-measure of D_2, the set of points $x \in Z_n$ whose fiber contains both H_x and V_x, is zero.

Consider D_H the set of points $x \in Z_n$ whose fiber contains only H_x but not V_x, and let D_V be the set of points in Z_n whose fiber contains V_x but not H_x. The results above show that $D_H \cup D_V$ has full η_n-measure. By 4.10, it follows that $g_n(D_H) = D_V$ and $g_n(D_V) = D_H$. Therefore,

$$\eta_n(D_H) = \eta_n(D_V) = \frac{1}{2}.$$

However, since g_n is Bernoulli, g_n^2 is also ergodic which is a contradiction because, $g_n^2(D_H) = D_H$ and $g_n^2(D_V) = D_V$.

4.2.3 Discontinuity example in $GL(2, \mathbb{R})^2$

Let $M = (GL(2, \mathbb{R}))^{\mathbb{Z}}$ let $f : M \to M$ be the shift map over M and $A : M \to GL(2, \mathbb{R})$ the product of random matrices. Now consider $X = GL(2, \mathbb{R})^2$ with the maximum norm, and let $N = X^{\mathbb{Z}}$ be the space of sequences over X and $g : N \to N$ the shift map over N. As before, we can identify N with M using the function $\iota : M \to N$ defined by $\iota((\alpha_n)_n) = (\beta_n)_n$ where $\beta_n = (\alpha_{2n}, \alpha_{2n+1})$ which is a bijection between N and M.

With the above definition we can see that

$$g(\iota((\alpha_n)_n)) = f^2((\alpha_n)_n$$

and defined $B : N \to GL(2,\mathbb{R})$ the linear cocycle induced by A in N by $B(\iota((\alpha_n)_n)) = A^2((\alpha_n)_n)$. In a similar way as the previous example there exist a measure p and a sequence of measures $(p_k)_k$ on X converging to p in the Warssestein topology, such that $\lambda_+(A, p_k) \nrightarrow \lambda_+(A, p)$.

Indeed, let $\alpha : \mathbb{N} \to GL(2,\mathbb{R})$ be defined by

$$\alpha(2k-1) = \begin{pmatrix} k & 0 \\ 0 & k^{-2} \end{pmatrix}$$

$$\alpha(2k) = \begin{pmatrix} k^{-2} & 0 \\ 0 & k \end{pmatrix}.$$

Taking $m \in \mathbb{N}$ the smallest natural (odd) number bigger than 3 such that $\sum_{n\geq m} e^{-\sqrt{n}}$ is less than 1, which exist since $\sum_k e^{-\sqrt{k}}$ is convergent, and define

$$\begin{aligned} p_{2k} &= p_{2k-1} = \frac{1}{2}e^{-\sqrt{k}}, \text{ if } 2k-1 \geq m, \\ p_3 &= 1 - \sum_{n\geq m} e^{-\sqrt{n}}, \\ p_k &= 0, \text{ otherwise.} \end{aligned}$$

and let $\tilde{q} = \sum_{k\in\mathbb{N}} p_k \delta_{\alpha(k)},$. Consider the space $\Omega = \mathbb{N}^2$ and define the measure on X by $q = \sum_{w\in\Omega} p_w \delta_{\alpha(w)}$, where $\alpha(w) = (\alpha(w_1), \alpha(w_2))$ and, $p_w = p_1 p_2$ if $w = (w_1, w_2)$. Let $\nu = q^{\mathbb{Z}}$ a measure on N.

Analysis similar to that in Section 4.2.2 shows that $q \in P_1(X)$, and using that $\nu = i_*\mu$ we have

$$\begin{aligned} \lambda_+(B, q) &= \lim_n \frac{1}{n} \int_N \log \|B^n(x)\| d\nu \\ &= \lim_n \frac{1}{n} \int_M \log \|A^{2n}(x)\| d\mu \\ &= 2\lambda_+(A, \tilde{q}) \\ &= 2p_3 \log 2 > 0. \end{aligned}$$

For each $n \in \mathbb{N}$ consider $w_n = (2n, 2n-1)$ and define $\beta(w_n) = (\beta(2n), \beta(2n-1))$, by

$$\beta(2n) = \begin{pmatrix} 1 & -\delta \\ 0 & 1 \end{pmatrix} \alpha(2n) \begin{pmatrix} 1 & 0 \\ \epsilon & 1 \end{pmatrix} = \begin{pmatrix} 0 & -n\delta \\ \epsilon n & n \end{pmatrix}$$

$$\beta(2n-1) = \begin{pmatrix} 1 & 0 \\ \epsilon & 1 \end{pmatrix} \alpha(2n-1) = \begin{pmatrix} n & 0 \\ \epsilon n & n^{-1} \end{pmatrix},$$

where $\delta = n^{-(1+\gamma)}$ with $0 < \gamma < 1$, $\epsilon = n^{-3}\delta^{-1} = n^{\gamma-2}$.

We proceed to define the sequence by

$$q_n = \sum_{w \neq w_n} p_w \delta_{\alpha(w)} + p_{w_n} \delta_{\beta(w_n)}.$$

To prove that $W(q_n, q) \to 0$ if n goes to infinite we consider the diagonal coupling of q_n and q

$$\pi_n = \sum_{w \neq w_n} p_w \delta_{(\alpha(w), \alpha(w))} + p_{w_n} \delta_{(\alpha(w_n), \beta(w_n))}$$

Hence, we have

$$\begin{aligned} W(q_n, q) &\leq \int d_\infty(u, v) \pi_n(u, v) \\ &= p_{w_n} d_\infty(\alpha(w_n), \beta(w_n)) \\ &< \max\{\|\beta(2n) - \alpha(2n)\|, \|\beta(2n-1) - \alpha(2n-1)\|\} \\ &\leq \max\{\epsilon\sigma_n, n^{-2} + n\delta\} \\ &= \max\{n^{\gamma-1}, n^{-2} + n^{-\gamma}\} \\ &\leq 2n^{-l} \end{aligned}$$

where $l = \min\{\gamma, 1-\gamma\} > 0$, which proofs our claim.

The rest of the proof, that is proving that $\lambda_+(B, q_n) = 0$ for all n, runs as before by noticing that for any $x \in Z_n = [0 : \beta(w_n)]$

$$B(x) = \begin{pmatrix} 0 & -n^2\delta \\ \epsilon n^{-1} & 0 \end{pmatrix}.$$

Indeed, this guarantees that $B(x)H_x = V_{g(x)}$ and $B(x)V_x = H_{g(x)}$ where $H_x = \mathbb{R}(1,0)$ and $V_x = \mathbb{R}(0,1)$. Finally, applying the argument of the first return map as in Section 4.2.2 we conclude our proof.