Quantitative and Extremal Problems in Graphs and Hypergraphs

by

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Abstract

In this thesis we investigate several extremal problems in graphs and hypergraphs. In graphs we study generalized Turán Problems in simple graphs and in planar graphs. In hypergraphs we study Turán numbers of Berge Graphs and the Ramsey numbers of certain families. The thesis is divided in five chapters.

In the first chapter the background needed on the different chapters is presented: Graph Theory, Planar Graphs, Extremal Graph Theory, Generalized Turán Numbers and Ramsey Theory.

In the second chapter we count the number of paths in a graph that does not contain a longer path, we find asymptotic and exact results in some cases. We also consider other structures like stars and the set of cycles of length at least k, where we derive asymptotically sharp estimates. Our results generalize well-known extremal theorems of Erdős and Gallai. These results are based on the paper "The maximum number of P_ℓ copies in P_k-free graphs" co-authored with Ervin Győri, Nika Salia and Casey Tompkins.

In the third chapter we consider a generalized Turán problem in planar graphs. Hakimi and Schmeichel considered the problem of maximizing the number of cycles of a given length in an n-vertex planar graph. They precisely determined the maximum number of triangles and 4-cycles and presented a conjecture for the maximum number of pentagons. We confirm their conjecture. Even more, we characterize the n-vertex, planar graphs with the maximum number of pentagons. These results are based on the paper "The Maximum Number of Pentagons in a Planar Graph" co-authored with Ervin Győri, Adissu Paulos, Nika Salia and Casey Tompkins.

In the fourth chapter we consider variants of a classical conjecture of Erdős and Sós, which asks to determine the Turán number of a tree. We study this problem in the settings of hypergraphs and multi-hypergraphs. In particular, for all k and r, with $r \geq k(k-2)$, we show that any r-uniform hypergraph $\mathcal{H}$ with more than $\frac{n(k-1)}{r+1}$ hyperedges contains a Berge copy of any tree with k edges different from the k-edge star. This bound is sharp when $r+1$ divides n and for such values of n we determine the extremal hypergraphs. These results are based on the paper "Turán numbers of Berge trees" co-authored with Ervin Győri, Nika Salia and Casey Tompkins.

In the fifth chapter we study Ramsey numbers of hypergraphs. In particular, we show that $R^3(BK_s, BK_t) = s + t - 3$ for $s, t \geq 4$ and $\max\{s, t\} \geq 5$ where BK_n is a Berge-K_n hypergraph. For higher uniformity, we show that $R^4(BK_t, BK_t) = t + 1$ for $t \geq 6$ and $R^k(BK_t, BK_t) = t$ for $k \geq 5$ and t sufficiently large. We also investigate the Ramsey number of trace hypergraphs, suspension hypergraphs and expansion hypergraphs. These results are based on the paper "Ramsey numbers of Berge-hypergraphs and related structures" co-authored with Nika Salia, Casey Tompkins and Zhiyu Wang.

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Chapter 1

Introduction

1.1 Graph Theory

A *graph* $G = (V(G), E(G))$ is a pair of sets, where $V(G)$ is a non-empty finite set, called the vertex set of G and E is a subset of $\{\{u, v\} : u, v \in V, u \neq v\}$, called the *vertex set edge set* of G. We denote the size of this sets by $v(G) = |V(G)|$ and $e(G) = \{E(G)\}$ the number of vertices and the number of edges of the graph.

If $v, u \in V$ are vertices such that $\{u, v\} \in E$, we say that u and v are *adjacent*, we say that the edge $\{u, v\}$ is incident with the vertices u and y. Given a set $S \subseteq V$ and an edge e, we say that e is incident with S if at least one of the vertices e is incident with it in S. Given $v \in V$, we define the *neighborhood* of v to be the set $N(v) := \{u \in V : \{v, u\} \in E\}$, and we define the *degree* of v as the number $d_G(v) = |N(v)|$, when the base graph is clear we simply denote the degree of v as $d(v)$. For a graph G we denote by $\delta(G)$ its *minimum degree*, that is the smallest possible value of $d(v)$ among the vertices of V.

Theorem 1.1. *For any graph G we have the identity*

$$\sum_{v \in V(G)} d(v) = 2e(G).$$

Definition 1.2. *For a graph G, we denote by $d(G)$ the average degree of G, that is $d(G) = \frac{2e(G)}{v(G)}$.*

Lemma 1.3. *Any non-empty graph G contains a subgraph G' with minimum degree greater than $d(G)/2$.*

The previous lemma is a well-known result in graph theory, which can be proved using the following lemma.

Lemma 1.4. *Let G be a graph and $V' \subseteq V$, if V' is incident with at most $\frac{d(G)}{2}|V'|$ edges, then $d(G[V \setminus V']) \geq d(G)$.*

A graph F is called a *subgraph* of G if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. Given a set $S \subseteq V(G)$, we denote by $G[S]$ the induced subgraph of G with vertex set S. A set is called independent if the graph induced by S has no edges. The *independence number* $\alpha(G)$ denotes the maximum size of an independent set in G.

Let F and G be graphs. If $f : V(F) \to V(G)$ is an injective function, such that, for any $x, y \in V(F)$ if $\{x, y\} \in E(F)$ then $\{f(x), f(y)\} \in E(G)$, then we call the subgraph $(f(V(F)), f(E(F))$ of G a *copy* of F. We say that G is F*-free* if here is no copy of F in G. We denote by $\mathcal{N}(F, G)$ the number of copies of F in G.

Definition 1.5. *A path in a graph is a sequence of distinct vertices $v_1, v_2, \dots, v_{t+1}$ such that v_i and v_{i+1} are adjacent for every $i = 1, 2, \dots, t$. The vertices x_1 and x_{t+1} are referred to as terminal vertices, and the remaining vertices are referred to as internal vertices.*

Definition 1.6. *A graph is connected if for every pair of vertices u, v there is a path starting from u and ending in v.*

Definition 1.7. *A cycle is a sequence $v_1, v_2, \dots, v_{k-1}, v_k = v_1$ where v_i and v_{i+1} are adjacent for $i = 1, 2, \dots, k-1$ and v_i is distinct to v_j for any $1 \le i < j \le k-1$.*

Definition 1.8. *A connected graph which has no cycles is called a tree.*

Theorem 1.9. *A connected graph G is a tree if and only if $e(G) = v(G) - 1$.*

We denote by P_k the path on k edges, by C_k the cycle on k vertices, and by K_r the *complete graph* on r vertices, that is, K_r is a graph on r vertices such that every pair of vertices is adjacent.

Definition 1.10. *A graph G is a bipartite graph if $V(G)$ can be partitioned into two color classes X and Y such that every edge of G contains precisely one vertex of each class.*

We denote by $K_{s,t}$ the *complete bipartite graph* with color classes of X and Y, with $|X| = s$, $|Y| = t$ and x is adjacent to y for every pair of vertices $x \in X, y \in Y$.

Definition 1.11. *A matching in a graph is a set of disjoint edges.*

An important result about matching in bipartite graphs is given by a Theorem of Hall [54].

Theorem 1.12 (Hall [54])**.** *Let G be a bipartite graph with color classes X and Y. If $|N(S)| \ge |S|$ for every $S \subseteq X$, then there exists a matching in G that covers every vertex of X.*

Corollary 1.13. *Let G be a bipartite graph with color classes X and Y. If there exists a number k such that $d(x) \ge k$ for every $x \in X$ and $d(y) \le k$ for every $y \in Y$, then there exists a matching that covers every vertex of X.*

One important and classical result in extremal graph theory is Dirac's Theorem [15].

Theorem 1.14 (Dirac [15])**.** *Let $n \ge 3$ and G be an n-vertex graph. If $\delta(G) \ge \frac{n}{2}$, then G contains an n-vertex cycle. i.e. a Hamiltonian cycle.*

1.2 Planar Graphs

A graph is said to be *planar*, if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing of a planar graph G is called a *planar embedding* of G.

Let $C = x_1, x_2, \dots x_k, x_1$ be a cycle in G, then C is said to *separate* vertices $y, z \in V(G)$ in a planar embedding of G if one of y or z is in the interior of the curve formed by embedding of the cycle and the other one is in the exterior.

One of the most basic results in planar graphs is Euler's Formula.

Theorem 1.15. *If G is a connected planar graph, then*

$$v(G) - e(G) + f = 2,$$

where f is the number of faces the planes is divided into in a planar embedding of G.

From Euler's formula we obtain three important corollaries.

Corollary 1.16. *Every planar embedding of a planar graph has the same number of faces.*

Corollary 1.17. *If G is a planar graph with $n \geq 3$ vertices, then $e(G) \leq 3n - 6$.*

An n-vertex planar graph G has $3n - 6$ edges if and only if in any planar embedding every face of G is a triangle, and so an n-vertex planar graph with $3n - 6$ edges is called a triangulation. A planar graph G is called maximal if it is not possible to add an extra edge to G and preserve the planarity. Moreover, if an n-vertex graph has less than $3n-6$ edges, then it is always possible to add another edge while keeping the graph planar. Hence the maximal planar graphs are precisely the triangulations.

Corollary 1.18. *If G is a planar graph, then $\delta(G) \leq 5$.*

It is known that K_5 and $K_{3,3}$ are not planar graphs, and therefore, a graph G can only be planar if it is K_5-free and $K_{3,3}$-free.

Theorem 1.19 (Kuratowski [61])**.** *A graph G is planar if and only if G does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.*

For a given graph G, if $e = \{v, u\}$ is an edge of G, then the *contraction* of the edge e in G is the graph obtained from G by replacing the two vertices $\{v, u\}$ with a new vertex w and replacing the edges of the form $\{v, x\}$ and $\{y, u\}$ with the edges $\{w, x\}$ and $\{y, w\}$ taking the new edges without multiplicity.

1.3 Extremal Graph Theory

Turán-type problems ask to determine the extremal values of a graph property in a family of graphs. One of the first results is due to Mantel [69].

Theorem 1.20 (Mantel [69])**.** *The maximum number of edges in an n-vertex triangle-free graph is $\lfloor \frac{n^2}{4} \rfloor$. Furthermore, the only triangle-free graph with $\lfloor \frac{n^2}{4} \rfloor$ edges is the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.*

A generalization of this result by Turán [83], is in fact what started the study of the extremal numbers, also known as Turán numbers. One of the main problems in extremal graph theory is to determine for a given graph H (or family of graphs) the Turán number $\mathrm{ex}(n, H)$, which denotes the maximum number of edges an H-free graph on n vertices.

For a graph G, the *chromatic number* $\chi(G)$ is the minimum number of colors needed to color the vertex set of G so that every pair of adjacent vertices has a different color assigned. It turns out that the chromatic number is a decisive factor in the asymptotic behavior of the Turán number.

Theorem 1.21 (Erdős-Stone-Simonovits [24, 25]). *For a graph H, we have*

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right)\binom{n}{2} + o(n^2).$$

In particular, Theorem 1.21 implies that the asymptotic value of $\mathrm{ex}(n, H)$ is determined, when $\chi(H) \geq 3$, while this Theorem only says that $\mathrm{ex}(n, H) = o(n^2)$ when $\chi(H) = 2$, that is if H is bipartite.

Kővári, Sós and Turán [60] considered the case when the forbidden graph is the complete bipartite graph $K_{s,t}$.

Theorem 1.22 (Kővári-Sós-Turán [60]). *For any natural numbers s and t, there exists a constant c such that*

$$\mathrm{ex}(n, K_{s,t}) \leq cn^{2-1/s}.$$

Another problem that has been studied is the Turán number of cycles.

Theorem 1.23 (Erdős [21], Bondy-Simonovits [10]). *For every integer k, there exists constants c_k and d_k such that*

$$c_k n^{1+\frac{1}{2k-1}} \leq \mathrm{ex}(n, C_{2k}) \leq d_k n^{1+\frac{1}{k}}.$$

As a consequence of Lemma 1.3 it follows that for a given tree T with k edges, $\mathrm{ex}(n, T) \leq n(k-1)$. This fact together with the previous theorem imply that $\mathrm{ex}(n, H) = O(n)$ if and only if H is a tree. For a path Erdős and Gallai [23] proved the following result

Theorem 1.24 (Erdős–Gallai [23]). *For all $n \geq k$,*

$$\mathrm{ex}(n, P_k) \leq \frac{(k-1)n}{2}.$$

Moreover, equality holds if and only if k divides n and G is the disjoint union of cliques of size k.

In their paper, Erdős and Gallai deduced Theorem 1.24 as a corollary of the following result about graphs with no long cycles.

Theorem 1.25 (Erdős–Gallai [23]). *For all $n \geq k$, for $C_{\geq k}$ the family of cycles of length at least k, we have*

$$\mathrm{ex}(n, C_{\geq k}) \leq \frac{(k-1)(n-1)}{2}.$$

Moreover, equality holds if and only if $k-2$ divides $n-1$ and G is a connected graph such that every block of G is a clique of size $k-1$.

Erdős and Sós [22] conjectured that the extremal number of any tree should be the same as the path, a conjecture that at this moment has only been proven for special families of trees.

Conjecture 1.26 (Erdős-Sós [22]). *Let T be a fix tree on k edges, then*

$$\mathrm{ex}(n, T) \leq \frac{n(k-1)}{2}.$$

Extremal problems have also been considered for host graphs other than K_n. Examples include the Zarankiewicz problem where the host graph is taken to be a complete bipartite graph, or extremal problems on the hypercube Q_n initiated by Erdős [19]. More recently, extremal problems have been considered where the host graph is taken to be a planar graph. For a given graph F, let us denote the maximum number of edges in an n-vertex F-free planar graph by $\mathrm{ex}_{\mathcal{P}}(n, F)$ (note that if $\mathcal{F}$ is the family of $K_{3,3}$ and K_5 subdivisions we have by Theorem 1.19 that $\mathrm{ex}_{\mathcal{P}}(n, F) = \mathrm{ex}(n, \mathcal{F} \cup \{F\})$).

This topic was initiated by Dowden in [16] who determined $\mathrm{ex}_{\mathcal{P}}(n, C_4)$ and $\mathrm{ex}_{\mathcal{P}}(n, C_5)$. A variety of other forbidden graphs F including stars, wheels and fans were considered by Lan, Shi and Song [63]. The case of theta graphs was considered in Lan, Shi and Song [64], and the case of short paths was considered by Lan and Shi in [62].

1.4 Generalized Turán Numbers

A generalized version of the Turán problem has been also studied. One problem solved by Erdős [18] (also by Zykov [86]), was to instead of determining the maximum number of edges in a K_t-free, to determine the maximum possible number of copies of K_s for a give $s < t$.

It is natural to consider what happens if instead of counting edges, we count copies of another graph. Alon and Shikhelman [3] recently initiated a systematic approach to this kind of problem; they introduced the notation $\mathrm{ex}(n, T, H)$ to denote the maximum number of copies of a given graph T among H-free graphs with n vertices. Another problem considered by Erdős was to determine the maximum number of pentagons in a triangle free graph, $\mathrm{ex}(n, C_5, C_3)$.

Conjecture 1.27 (Erdős [18]). *For a positive integer n, we have that*

$$\mathrm{ex}(n, C_5, C_3) \leq (n/5)^5$$

and the graph which achieves the maximum is obtained by blowing up a 5-cycle.

Győri [45] obtained an upper bound of roughly $1.03(\frac{n}{5})^5$, and later Hatami, Hladký, Král, Norine and Razborov [55] and independently Grzesik [41] finally gave a positive answer to this conjecture.

Alon and Shikhelman [3] considered the problem of maximizing the number of copies of a tree T in a graph which is H-free, for another tree H. Given two trees T and H, they introduced an integer parameter $m(T, H)$ and proved that $\mathrm{ex}(n, T, H) = \Theta(n^{m(T,H)})$, thereby determining the correct order of magnitude for all pairs of trees. A recent result due to Letzter [66] extends the above result of Alon and Shikhelman to the case when only H is a tree and T is arbitrary. It is shown that, nonetheless, the order of magnitude of $\mathrm{ex}(n, T, H)$ is a positive integer power of n.

Another direction of research which has been considered is maximizing the number of copies of a given graph in an n-vertex planar graph. Hakimi and Schmeichel [53] determined the maximum number of triangle and C_4 copies possible in a planar graph. In this setting Alon and Caro [2] determined the maximum number of copies of $K_{1,t}$, $K_{2,t}$ and K_4 possible. Resolving a conjecture attributed to Perles [2], Wormald [85] proved that every 3-connected graph H occurs at most $c_H n$ times in an n-vertex planar graph for some constant c_H depending on H (this result was proved again in a different way by Eppstein [17]). A simple argument shows that graphs with at least 3 vertices which are at most 2-connected will occur at least quadratically many times in a planar graph. Thus, the preceding result of Wormald [85] and Eppstein [17] provides a characterization of graphs which can occur at most $O(n)$ times in a planar graph. The problem of maximizing the total number of cliques in such graphs was investigated in a series of papers culminating in [65] and [29].

It is interesting to note that the problem of maximizing H copies in a planar graph is in some sense a special case of a generalized Turán Problem. Indeed, for a given graph H, and the collection $\mathcal{F}$ of subdivisions of $K_{3,3}$ and K_5, it follows from Theorem 1.19 theorem that $\mathrm{ex}(n, H, \mathcal{F})$ is equal to the maximum number of H-copies in an n-vertex planar graph.

1.5 Extremal Hypergraph Theory

A hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ is pair, where $V(\mathcal{H})$ is a non-empty finite set, called the *vertex sett* of $\mathcal{H}$, and $E(\mathcal{H})$ is a subset of the power set of $V(\mathcal{H})$, called the *edge set* of G. We denote the size of these sets by $v(G) = |V(G)|$ and $e(G) = |E(G)|$ the number of vertices and the number of edges of the graph. We say that a hypergraph is r-uniform if every hyperedge has size r. By $K_t^{(r)}$ we denote the t-vertex r-uniform clique, the hypergraph consisting on t vertices and containing every possible r-set as a hyperedge.

For a hypergraph $\mathcal{H}$, the *incidence bipartite graph* of $\mathcal{H}$ is the bipartite graph G with color classes $V(\mathcal{H})$ and $E(\mathcal{H})$ such that there is an edge between $v \in V(\mathcal{H})$ and $h \in E(\mathcal{H})$ if and only if $v \in h$.

As in the graph case, there is a notion of the Turán number for a family of hypergraphs.

Definition 1.28. *The Turán number of a family of r-uniform hypergraphs $\mathcal{F}$, denoted $\mathrm{ex}_r(n, \mathcal{F})$, is the maximum number of hyperedges in an n-vertex, r-uniform, simple-hypergraph which does not contain an isomorphic copy of $\mathcal{H}$, for all $\mathcal{H} \in \mathcal{F}$, as a sub-hypergraph.*

However for hypergraphs the problem of determining $\mathrm{ex}_r(n, F)$ for an r-uniform hypergraph is much less understood that the graph case (see [58] for a survey). A particular family of hypergraphs for which the Turán number has been recently studied is the family of Berge hypergraphs.

Berge [8] presented the following definition for paths and cycles in hypergraphs.

Definition 1.29. *A* Berge path *of length t in a hypergraph is an alternating sequence of distinct vertices and hyperedges, $v_0, e_1, v_1, e_2, v_2, \ldots, e_t, v_t$ such that, $v_{i-1}, v_i \in e_i$, for $i = 1, 2, \ldots, t$. The vertices v_i are called defining vertices and the hyperedges e_i are called defining hyperedges.*

Definition 1.30. *A* Berge cycle *of length t in a hypergraph is an alternating sequence of distinct vertices and hyperedges, $v_0, e_1, v_1, e_2, v_2, \ldots, v_{t-1}, e_t, v_0$ such that, $v_{i-1}, v_i \in e_i$, for $i = 1, 2, \ldots, t$, (where indices are taken modulo t). The vertices v_i are called defining vertices and the hyperedges e_i are called defining hyperedges.*

In [46] Győri started the study of Berge cycles where he determined that the maximum number of hyperedges in a 3-uniform hypergraph that contains no Berge cycle of length 3 is at most $\frac{n^2}{8}$. The maximum number of hyperedges in an r-uniform graphs with no Berge path of length k was determined by Győri, Katona and Lemons [47] (one of the cases was done later by Davoodi, Győri, Methuku and and Tompkins [14]).

Based on the previous definition Gerbner and Palmer [38] introduced a a general notion of a Berge copy of G in a hypergraph, for any graph G.

Definition 1.31. *Given a fixed graph G, a hypergraph $\mathcal{H}$ is a* Berge copy *of G, if there exists an injection $f_1 : V(G) \to V(\mathcal{H})$ and a bijection $f_2 : E(G) \to E(\mathcal{H})$, such that if $e = \{v_1, v_2\} \in E(G)$, then $\{f_1(v_1), f_1(v_2)\} \subseteq f_2(e)$.*

The set of Berge copies of G is denoted by $\mathcal{B}G$. The sets $f_1(V(G))$ and $f_2(E(G))$ are called the defining *vertices and hyperedges, respectively.*

Since its introduction, the Turán problem for Berge-G-free hypergraphs has been investigated heavily (see, for example [5], [40] and [76]). Complete graphs were considered by several authors in [34], [35], [42], and [68].

1.6 Ramsey Theory

Ramsey theory is among the oldest and most intensely investigated topics in combinatorics. It began with the seminal result of Ramsey from 1930.

Theorem 1.32 (Ramsey [78]). *Let r, t and k be positive integers. Then there exists an integer N such that any coloring of the N-vertex r-uniform complete hypergraph with k colors contains a monochromatic copy of the t-vertex r-uniform complete hypergraph.*

The statement of Ramsey's Theorem extends to any families of hypergraphs.

Definition 1.33. *Let $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_k$ be nonempty collections of r-uniform hypergraphs. there exists an integer N such that if the hyperedges of the complete r-uniform N-vertex hypergraph are colored with k colors, then for some $1 \leq i \leq k$, there is a monochromatic copy of a member of $\mathcal{H}_i$. We denote by $R_k^r(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_k)$ the minimum such N. If k is clear by context, then we omit k in this notation. If a collections $\mathcal{H}_i$ consist of a single hypergraph $\mathcal{G}$, then we write $\mathcal{G}$ in place of $\mathcal{H}_i = \{\mathcal{G}\}$.*

Estimating the smallest value of such an integer N (the so-called Ramsey number) is a notoriously difficult problem and usually only weak bounds are known, for the so-called diagonal Ramsey number of graphs we know the following bounds

Theorem 1.34 (Erdős-Szekeres [26], Erdős [20]). *For every $n > 2$, we have*

$$2^{\frac{n}{2}} \leq R^2(K_n, K_n) \leq 2^{2n}.$$

Given the difficulty of this problem, many people began investigating variations of this problem where graphs other than the complete graphs are considered. An example of an early result in this direction due to Chvátal [11] asserts that the Ramsey number of a t-clique versus any m-vertex tree is precisely $N = 1 + (m-1)(t-1)$. That is, any red-blue coloring of the complete graph K_N yields a red K_t or a blue copy of a given m-vertex tree.

Ramsey problems for a variety of hypergraphs and classes of hypergraphs have been considered (for a recent survey of such problems see [71]).

The Ramsey problem for Berge-paths and cycles has received much attention. Of particular interest is a result of Gyárfás and Sárközy [44] showing that the 3-color Ramsey number of a 3-uniform Berge-cycle of length n is asymptotic to $\frac{5n}{4}$ (the 2-color case was settled exactly in [43]).

Chapter 2

The maximum number of P_ℓ copies in P_k-free graphs

2.1 Introduction

Recall that we denote the path with k edges by P_k and the cycle with k edges by C_k. By $C_{\geq k}$ we mean the set of all cycles of length at least k. By S_k we denote the star on $k+1$ vertices.

We begin by recalling the theorem of Erdős and Gallai on P_k-free graphs (Theorem 1.24) as well as some recent generalizations due to Luo [67], where the number of cliques is considered.

Theorem (Erdős–Gallai [23])**.** *For all $n \geq k$,*

$$\mathrm{ex}(n, P_k) \leq \frac{(k-1)n}{2},$$

Moreover, equality holds if and only if k divides n and G is the disjoint union of cliques of size k.

As the extremal examples for Theorem 1.24 are disconnected, it is natural to consider a version of the problem where the base graph is assumed to be connected. Kopylov [59] settled this problem, and later Ballister, Győri, Lehel and Schelp [7] classified the extremal cases. Before stating this result, we will need the following definition.

Definition 2.1. *We denote by $G_{n,k,a}$ the graph whose vertex set is partitioned into 3 classes, A, B and C with $|A| = a$, $|B| = n-k+a$, $|C| = k-2a$ such that $A \cup C$ induces a clique, B is an independent set and all possible edges are taken between vertices of A and B. (See Figure 2.1.)*

Throughout this section we let $t = \left\lfloor \frac{k-1}{2} \right\rfloor$. In $G_{n,k,t}$, the class C has one vertex when k is odd or two vertices when k is even. By grouping B and C together, we have that $G_{n,k,t}$ is obtained from a complete bipartite graph $K_{t,n-t}$ by adding all edges in the color class of size t, and in the case when k is even, adding one additional edge inside the color class of size $n-t$.

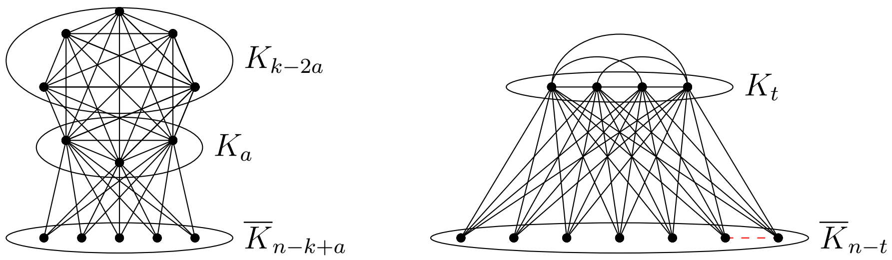


Figure 2.1: The graph $G_{n,k,a}$ is pictured on the left, and the special case of $G_{n,k,t}$ is pictured on the right. The dashed edge appears only when k is even.

Theorem 2.2 (Kopylov [59], Ballister–Győri–Lehel–Schelp [7]). *Let G be a connected n-vertex P_k-free graph, with $n \geq k$, then*

$$e(G) \leq \max(e(G_{n,k,t}), e(G_{n,k,1})).$$

Moreover, the extremal graph is either $G_{n,k,t}$ or $G_{n,k,1}$.

We have that

$$e(G_{n,k,1}) = \binom{k-1}{2} + n - k + 1 \quad \text{and} \quad e(G_{n,k,t}) = t(n-t) + \binom{t}{2} + \eta_k,$$

where η_k is 1, if k is even, and 0 otherwise. Therefore, the maximum in Theorem 2.2 is achieved by $G_{n,k,t}$ when $n \geq 5k/4$.

The following theorem was deduced by Luo [67] as a corollary of her main result. This result also follows from Theorem 1.24 using a simple induction argument. We present this proof here.

Theorem 2.3 (Luo [67]).

$$\mathrm{ex}(n, K_r, P_k) \leq \frac{n}{k}\binom{k}{r}.$$

Proof. We use induction on r, and the base case $r = 2$ is Theorem 1.24. Let G be an n-vertex graph containing no P_k. We have

$$\begin{aligned} r\mathcal{N}(K_r, G) &= \sum_{v \in V(G)} \mathcal{N}(K_{r-1}, G[N(v)]) \\ &\leq \sum_{v \in V(G)} \frac{v(G[N(v)])}{k-1}\binom{k-1}{r-1} = \frac{r}{k(k-1)}\binom{k}{r} 2e(G), \end{aligned}$$

since $G[N(v)]$ contains no P_{k-1}. By Theorem 1.24, we have $e(G) \leq \frac{(k-1)n}{2}$, and the result follows. $\square$

For our results we will need only that $\mathrm{ex}(n, K_r, P_k) \leq c_{k,r}n$ for some constant $c_{k,r}$ depending only on k and r.

If we impose the additional condition that the graph is connected, then the situation is more complicated. Luo proved the following sharp bounds.

Theorem 2.4 (Luo [67]). *Let $n > k \geq 3$ and G be a connected n-vertex graph with no path of length k, then*

$$\mathcal{N}(K_r, G) \leq \max\left(\mathcal{N}(K_r, G_{n,k,t}), \mathcal{N}(K_r, G_{n,k,1})\right).$$

Theorem 2.5 (Luo [67]). *Let $n \geq k \geq 4$ and G be a n-vertex graph with no cycle of length k or greater, then*

$$\mathcal{N}(K_r, G) \leq \frac{n-1}{k-2}\binom{k-1}{r}.$$

Some recent generalizations of the Erdős–Gallai theorem and Luo's results can be found in [75]. The results in this chapter focus on the case where sufficiently long paths or all sufficiently long cycles are forbidden. The general problem of enumerating cycles of a fixed length when a fixed cycle is forbidden has also been considered recently (see [39] and [35] which generalized earlier results for special cases, e.g., [9], [49], [3]).

In this chapter, we are interested in the case where the forbidden graph is a path. We find asymptotic values and sometimes the exact bound for the maximum number of copies of a smaller path (as well as for several other types of graphs). We also obtain asymptotic results for the problem of maximizing copies of T in a graph with no cycles of length at least k, in the case when T is a path.

This Chapter is organized as follows: In Section 2.2, we determine asymptotically the maximum number of paths and cycles in a P_k-free graph. For the case when k is even we provide a simple proof using a result of Nikiforov [74] on the spectral radius of P_k-free graphs. Then, we give more precise estimates, which are also sharp in the case when k is odd, through double-counting arguments. In Section 2.3, we determine the order of magnitude of $\mathrm{ex}(n, H, T)$ when T is a tree for the class of graphs H which satisfy the condition that $v(H) - \alpha(H) \leq \left\lfloor \frac{k-1}{2} \right\rfloor$. In Section 2.4, we determine $\mathrm{ex}(n, H, P_k)$ exactly for several graphs H including 4-cycles, stars and short paths. In Section 2.5, we consider the problem of enumerating copies of P_{k-1} in a P_k-free graph. We determine an asymptotic result for copies of P_5 in a P_6-free graph and pose a general conjecture.

2.2 Asymptotic Results

We write $f(n,k) \sim g(n,k)$ when $\lim_{k\to\infty}\left(\lim_{n\to\infty}\frac{f(n,k)}{g(n,k)}\right) = 1$. We estimate the number of copies of paths and cycles in a P_k-free graph. For a fixed $\ell \in \mathbb{N}$, we prove the following asymptotic results:

Theorem 2.6 (Győri, Salia, Tompkins, Zamora. [51]).

$$\mathrm{ex}(n, P_{2\ell}, P_k) \sim \frac{k^\ell n^{\ell+1}}{2^{\ell+1}}.$$

Theorem 2.7 (Győri, Salia, Tompkins, Zamora. [51]).

$$\mathrm{ex}(n, P_{2\ell+1}, P_k) \sim \frac{(\ell+2)k^{\ell+1}n^{\ell+1}}{2^{\ell+2}}.$$

Theorem 2.8 (Győri, Salia, Tompkins, Zamora. [51])**.**

$$\mathrm{ex}(n, C_{2\ell}, P_k) \sim \frac{k^\ell n^\ell}{\ell 2^{\ell+1}}.$$

Theorem 2.9 (Győri, Salia, Tompkins, Zamora. [51])**.**

$$\mathrm{ex}(n, C_{2\ell+1}, P_k) \sim \frac{k^{\ell+1} n^\ell}{2^{\ell+2}}.$$

The construction showing the lower bounds for Theorems 2.6 through 2.9 is the same as the extremal construction for the connected version of the Erdős–Gallai theorem, Theorem 2.2. Because we are interested in asymptotics, we will omit the edge from this construction which only occurs when k is even. Our n-vertex graph G is defined by taking a clique on a set S of $\lfloor \frac{k-1}{2} \rfloor$ vertices and connecting every vertex in S to every vertex of an independent set U on $n-t$ vertices. It is easy to see that this graph is P_k-free. In enumerating the copies of $P_{2\ell}$, the only paths which contribute asymptotically alternate between S and U, starting and ending with U (the factor of 2 comes from counting the path in both directions).

When enumerating the copies of $P_{2\ell+1}$, we have two kinds of paths which contribute asymptotically: those that start and end in U, using an edge in S at some step, and those that start in U and end in S, never using an edge contained in S. For the first type, we condition on which step in the path we use the edge in S (ℓ possibilities). Each such path gets counted twice, hence we divide by two. For the second type, each path is counted once and so we do not have to divide by 2.

Recall that the spectral radius of a graph G is the maximum of the eigenvalues of the adjacency matrix of G. We begin by showing how Theorem 2.6 can be derived from a result about the spectral radius of P_k-free graphs due to Nikiforov [74]. He determined, for sufficiently large n, the maximal spectral radius of a P_k-free graph on n vertices. We are interested in asymptotics so we will make use of the following corollary which follows directly from the results in [74].

Corollary 2.10 (Nikiforov [74])**.** *If n is sufficiently large and G is a P_k-free graph, then the spectral radius of G is at most $\sqrt{\lfloor (k+1)/2 \rfloor n}$.*

Spectral proof of Theorem 2.6. Let G be a P_k-free graph on n vertices (for n large enough to satisfy Corollary 2.10). Let A be the adjacency matrix of G, then we have

$$2 \cdot \frac{\mathcal{N}(P_{2\ell}, G)}{n} \leq \frac{\#\{2\ell\text{-walks in } G\}}{n} = \frac{\mathbf{1}^t A^{2\ell} \mathbf{1}}{\mathbf{1}^t \mathbf{1}} \leq \left(\sqrt{\lfloor (k+1)/2 \rfloor n} \right)^{2\ell} = (\lfloor (k+1)/2 \rfloor n)^\ell.$$

Where $\mathbf{1}$ is the all 1's vector, and the second inequality comes from the fact that the spectral radius of a Hermitian matrix M is the supremum of the quotient $\frac{x^* M x}{x^* x}$, where x ranges over $\mathbb{C}^n \setminus \{0\}$. Therefore, for every $k \in \mathbb{N}$ and n sufficiently large we have $\mathrm{ex}(n, P_{2\ell}, k) \leq n^{\ell+1} \lfloor (k+1)/2 \rfloor^\ell / 2$. □

Unfortunately, it does not seem like this approach can be used to prove Theorem 2.7 as the bound it would yield is off by a factor of order $\sqrt{n}$.

We will now prove the upper bounds from which Theorems 2.6 and 2.7 are immediate consequences. We note that the upper bound we obtain for the $P_{2\ell}$-case is sharper than the bound given by using the spectral radius.

Proposition 2.11. *Let ℓ, k be positive integers with $2\ell < k$, then*

$$\mathrm{ex}(n, P_{2\ell}, P_k) \leq \frac{k^\ell n^{\ell+1}}{2^{\ell+1}} + O(n^\ell).$$

Proposition 2.12. *Let ℓ, k be positive integers with $2\ell + 1 < k$, then*

$$\mathrm{ex}(n, P_{2\ell+1}, P_k) \leq \frac{(\ell+2)k^{\ell+1}n^{\ell+1}}{2^{\ell+2}} + O(n^\ell).$$

The proofs of the propositions above will use a double-counting argument involving structures defined using matchings. We will begin by estimating the maximum number of certain kinds of matchings occurring in a P_k-free graph.

Let us define M_1^ℓ, M_2^ℓ and M_3^ℓ to be the following graphs: M_1^ℓ is an $(\ell-1)$-matching together with a disjoint triangle, M_2^ℓ is an $(\ell-1)$-matching together with a disjoint K_4 and M_3^ℓ is an $(\ell-2)$-matching with two independent triangles, disjoint from the matching (see Figure 2.2).

Lemma 2.13. *Let $k, \ell \in \mathbb{N}$. The number of copies of M_1^ℓ, M_2^ℓ and M_3^ℓ in an n-vertex P_k-free graph is $O(n^\ell)$.*

Proof. Let G be a P_k-free graph on n vertices. By Theorem 2.3, the number of triangles in G is $O(n)$. By Theorem 1.24 the total number of edges in G is at most $(k-1)n/2$. It follows that the number of copies of M_1^ℓ is bounded from above by

$$\binom{\frac{kn}{2}}{\ell-1} O(n) = O(n^\ell).$$

The proofs of the bound for M_2^ℓ and M_3^ℓ are similar. □

Proof of Theorem 2.6. Let G be a P_k-free graph on n vertices. We will consider structures consisting of a matching of ℓ edges and a vertex not contained in these edges. Namely, a *matching structure* is an $(\ell+1)$-tuple $(e_1, e_2, \ldots, e_\ell, v)$ where $\{e_1, e_2, \ldots, e_\ell\}$ is a matching in G and $v \in V(G) \setminus \cup_{i=1}^{\ell} e_i$. We say that a path $P_{2\ell}$ *aligns* with a matching structure $(e_1, e_2, \ldots, e_\ell, v)$ if its edges are (consecutively) $e_1, f_1, e_2, f_2, \ldots, e_\ell, f_\ell$ where $v \in f_\ell$. We say that the matching structure *spans* the set of vertices $\cup_{i=1}^{\ell} e_i \cup \{v\}$.

Let $\mathcal{A} \coloneqq \{S \subseteq V : |S| = 2\ell+1, M_1^\ell \subseteq G[S]\}$. By Lemma 2.13, we have $|\mathcal{A}| = O(n^\ell)$. Let $\mathcal{M}$ be the set of all the matching structures which span a set of vertices not contained in $\mathcal{A}$. Then we have the following.

Claim 2.14. *At most one $P_{2\ell}$ aligns with each matching structure in $\mathcal{M}$.*

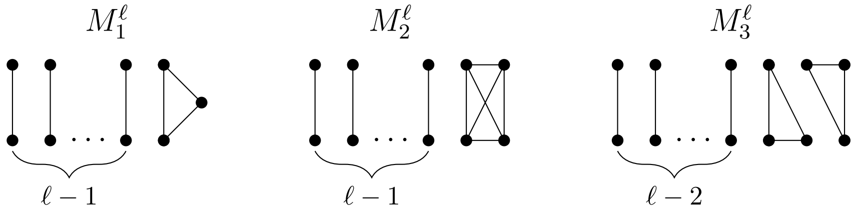


Figure 2.2: Matching structures with negligible contribution.

Proof. Let $(e_1, e_2, \ldots, e_\ell, v)$ be a matching structure in $\mathcal{M}$ and fix a $P_{2\ell}$ which aligns with it, say $a_1, b_1, a_2, b_2 \ldots, a_\ell, b_\ell, a_{\ell+1}$, where $e_i = \{a_i, b_i\}$ and $v = a_{\ell+1}$. Note that there is no edge from a_i to a_{i+1}, since otherwise $e_1, e_2, \ldots, e_{i-1}, \{b_{i+1}, a_{i+2}\}, \{b_{i+2}, a_{i+3}\}, \ldots, \{b_\ell, a_{\ell+1}\}$ together with the triangle $\{a_i, b_i, a_{i+1}\}$ forms an M_1^ℓ. Since there is a unique $P_{2\ell}$ spanning the matching structure and not containing an edge $\{a_i, a_{i+1}\}$, the claim is proved. (See Figure 2.3.) $\square$

Next, we observe that for every $P_{2\ell}$, there are precisely two matching structures for which that $P_{2\ell}$ is aligned. Indeed, let the vertices of the $P_{2\ell}$ be traversed in the order $v_1, v_2, \ldots, v_{2\ell+1}$, then the two matching structures with which the $P_{2\ell}$ aligns are

$$(\{v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{2\ell-1}, v_{2\ell}\}, v_{2\ell+1}) \text{ and } (\{v_{2\ell+1}, v_{2\ell}\}, \{v_{2\ell-1}, v_{2\ell-2}\}, \ldots, \{v_3, v_2\}, v_1).$$

It follows that the if we define $M := |\mathcal{M}|$, then the number of copies of $P_{2\ell}$ is bounded from above by $M/2 + O(n^\ell)$.

By Theorem 1.24, the number of edges in G is at most $(k-1)n/2$. A matching structure is formed by choosing ℓ edges in order followed by an additional vertex. Thus, we have the following upper bound on the number of matching structures in $\mathcal{M}$:

$$M \le \binom{\frac{nk}{2}}{\ell} \ell!\, n \le \frac{n^{\ell+1} k^\ell}{2^\ell}.$$

Dividing by 2 yields the required bound on the number of copies of $P_{2\ell}$. $\square$

Proof of Theorem 2.7. We will now define matching structures in a slightly different way. A *matching structure* is an $(\ell+1)$-tuple $(e_1, e_2, \ldots, e_{\ell+1})$, where $\{e_1, e_2, \ldots, e_{\ell+1}\}$ is a matching in G. A path $P_{2\ell+1}$ *aligns* with a matching structure $(e_1, e_2, \ldots, e_{\ell+1})$ if its edges are $e_1, f_1, e_2, f_2, \ldots, e_\ell, f_\ell, e_{\ell+1}$, consecutively.

Let $\mathcal{B} := \{S \subseteq V : |S| = 2\ell+2, M_2 \subseteq G[S]\}$ and $\mathcal{C} := \{S \subseteq V : |S| = 2\ell+2, M_3 \subseteq G[S]\}$. By Lemma 2.13, we have $|\mathcal{B}| = O(n^\ell)$ and $|\mathcal{C}| = O(n^\ell)$. Let $\mathcal{M}$ be the set of matching structures which do not span a vertex set in $\mathcal{B}$ or $\mathcal{C}$.

Claim 2.15. *There are at most $\ell + 2$ copies of $P_{2\ell+1}$ which align with each matching structure in $\mathcal{M}$.*

Proof. Consider a matching structure $(e_1, e_2, \ldots, e_{\ell+1}) \in \mathcal{M}$. We will consider the edges in the matching structure one by one and show that we can label the vertices of each edge e_j with a_j and b_j in such a way that there is no edge between a_j and a_{j+1}. Thus, every path which aligns with the matching structure will be a subgraph of the graph pictured (on the top) in Figure 2.4. Given that the matching structure has this form, we may easily upper bound the number of copies of $P_{2\ell+1}$ which can align with it. Indeed, if the

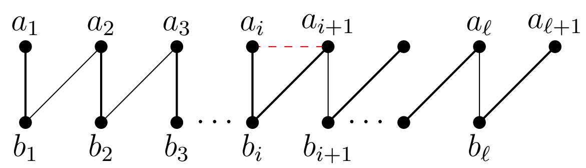


Figure 2.3: Matching structure from the proof of Claim 2.14.

$P_{2\ell+1}$ starts with the vertex b_1, there is at most one such path: $b_1, a_1, b_2, a_2, \ldots, b_{\ell+1}, a_{\ell+1}$. If it starts with the vertex a_1, then for at most one i, $1 \le i \le \ell$, the path may use an edge $\{b_i, b_{i+1}\}$; all other choices are forced. Thus, in total there are at most $1 + (\ell+1) = \ell+2$ paths which align with such a matching structure. We now prove that the desired labeling of the edges exists.

We may suppose that there is at least one edge from e_i to e_{i+1} for all $i = 1, 2, \ldots, \ell$, otherwise no $P_{2\ell+1}$ aligns with the matching structure. We also know $e_i \cup e_{i+1}$ does not induce a K_4, so there is at least one edge missing among these 4 vertices. Now we may label $e_1 = \{a_1, b_1\}$ in such a way that there is at least one edge missing from a_1 to e_2. Label $e_2 = \{a_2, b_2\}$ such that there is no edge between a_1 and a_2. In general, suppose we have already labeled the edges $e_1, e_2, \ldots, e_j$ in such a way that for $i \in \{1, 2, \ldots, j-1\}$, a_i is not adjacent to a_{i+1}. We will show that e_{j+1} can be labeled by a_{j+1} and b_{j+1} such that there is no edge between a_j and a_{j+1} or that we may be able to relabel the previous edges to achieve this.

We know that there is an edge missing from e_j to e_{j+1}. If there is an edge missing between a_j and e_{j+1}, then label $e_{j+1} = \{a_{j+1}, b_{j+1}\}$ so that there is an edge from a_j to a_{j+1}. Otherwise $\{a_j\} \cup e_{j+1}$ forms a triangle. In this case, there is an edge missing from b_j to e_{j+1}; label $e_{j+1} = \{a_{j+1}, b_{j+1}\}$ so that b_j is not adjacent to b_{j+1}.

Now if we do not have an edge from a_{j-1} to b_j, then we switch the labels on e_j and e_{j+1}, and we are done. (By switching the labels we mean that the vertex in e_i previously labeled a_i is now labeled b_i, and the vertex previously labeled b_i is now labeled a_i.) Thus, assume we have an edge from a_{j-1} to b_j. Then we have no edge from b_{j-1} to b_j, for this would yield an M_3. Next, consider e_{j-2}. If there is no edge from a_{j-2} to b_{j-1}, then switch the labels on e_{j-1}, e_j and e_{j+1}, and we are done. If there is an edge from a_{j-2} to b_{j-1}, we proceed similarly with e_{j-3}. Continuing this procedure, we will reach an edge e_r such that switching the labels of $e_r, e_{r+1}, \ldots, e_{j+1}$ yields no edge between a_i and a_{i+1} for any $1 \le i \le j$. (This procedure is illustrated in Figure 2.4.) $\square$

We now complete the proof of Theorem 2.7. Again we set $M := |\mathcal{M}|$. By Theorem 1.24, there are at most $(k-1)n/2$ total edges in G. Thus,

$$|\mathcal{M}| \le \binom{\frac{nk}{2}}{\ell+1}(\ell+1)! \le \frac{k^{\ell+1}n^{\ell+1}}{2^{\ell+1}}.$$

Since at most $\ell+2$ paths $P_{2\ell+1}$ align with each matching structure from $\mathcal{M}$, and every $P_{2\ell+1}$ aligns with precisely two matching structures. It follows that the total number of

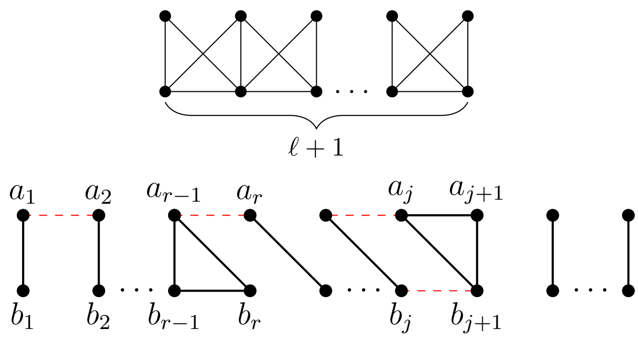


Figure 2.4: The structure of paths aligning with matching structures from $\mathcal{M}$.

copies of $P_{2\ell+1}$ in G is at most

$$\frac{(\ell+2)M}{2}+O(n^\ell)\le\frac{(\ell+2)\binom{\frac{nk}{2}}{\ell+1}(\ell+1)!}{2}+O(n^\ell)=\frac{(\ell+2)k^{\ell+1}n^{\ell+1}}{2^{\ell+2}}+O(n^\ell). \qquad \square$$

The lower bound for Theorems 2.8 and 2.9 also comes from $G_{n,k,t}$. Similarly as before, the upper bounds a consequence of the following propositions.

Proposition 2.16. *Let $2\ell<k$, then*

$$\mathrm{ex}(n,C_{2\ell},P_k)\le\frac{k^\ell n^\ell}{\ell 2^{\ell+1}}+O(n^{\ell-1}).$$

Proposition 2.17. *Let $2\ell+1<k$, then*

$$\mathrm{ex}(n,C_{2\ell+1},P_k)\le\frac{k^{\ell+1}n^\ell}{2^{\ell+2}}+O(n^{\ell-1}).$$

It is enough to prove the following claims from which the propositions above follow (a proof of this implication is included after the proof of the claim). Their proofs are similar, so we just give the proof of the first claim.

Claim 2.18. *For every $k,\ell\in\mathbb{N}$, there exists $n_0\in\mathbb{N}$ such that if $n\ge n_0$,*

$$\mathrm{ex}(n+1,C_{2\ell},P_k)-\mathrm{ex}(n,C_{2\ell},P_k)\le\frac{k^\ell n^{\ell-1}}{2^{\ell+1}}+O(n^{\ell-2}).$$

Claim 2.19. *For every $k,\ell\in\mathbb{N}$, there exists $n_0\in\mathbb{N}$ such that if $n\ge n_0$,*

$$\mathrm{ex}(n+1,C_{2\ell+1},P_k)-\mathrm{ex}(n,C_{2\ell+1},P_k)\le\frac{\ell k^{\ell+1}n^{\ell-1}}{2^{\ell+2}}+O(n^{\ell-2}).$$

To prove the claims we are going to use the following Lemma 2.20, which follows from similar methods as Theorem 1.14

Lemma 2.20. *Let G be a graph with minimum $\delta(G)>\frac{k-1}{2}$, then either each connected component of G has size at most k or G contains a path of length k.*

Proof of Claim 2.18. Let G be a P_k-free graph on $n+1$ vertices with maximum number of copies of $C_{2\ell}$. If $\delta(G)>t$, then by Lemma 2.20, every connected component must have size at most k, and then $\mathcal{N}(C_{2\ell},G)\le k^{2\ell-1}n$.

So assume $\delta(G)\le t$, and let v be a vertex of minimum degree. Then every $C_{2\ell}$ using v can be divided into two paths: v together with the vertex preceding it and following it in the cycle (forming a P_2), and the remaining $2\ell-3$ vertices (forming a $P_{2(\ell-2)}$). Note that every P_2 and $P_{2(\ell-2)}$ can be joined in at most two ways to make a $C_{2\ell}$. Therefore, the number of copies of $C_{2\ell}$ containing v is at most

$$2\binom{d(v)}{2}\mathrm{ex}(n,P_{2(\ell-2)},P_k)\le 2\binom{t}{2}\frac{k^{\ell-2}n^{\ell-1}}{2^{\ell-1}}+O(n^{\ell-2})\le\frac{k^\ell n^{\ell-1}}{2^{\ell+1}}+O(n^{\ell-2}). \qquad \square$$

We include a proof that Proposition 2.16 follows from Claim 2.18.

Proof that Claim 2.18 implies Proposition 2.16. We have

$$\begin{aligned}
\mathrm{ex}(n, C_{2\ell}, P_k) &= \mathrm{ex}(n_0, C_{2\ell+1}, P_k) + \sum_{s=n_0+1}^{n} (\mathrm{ex}(s, C_{2\ell+1}, P_k) - \mathrm{ex}(s-1, C_{2\ell+1})) \\
&\le \mathrm{ex}(n_0, C_{2\ell+1}, P_k) + \frac{k^\ell}{2^{\ell+1}} \sum_{s=1}^{n} \left(s^{\ell-1} + O(n^{\ell-2})\right) \\
&\le \mathrm{ex}(n_0, C_{2\ell+1}, P_k) + \frac{k^\ell}{2^{\ell+1}} \sum_{s=1}^{n} \left(\frac{(s+1)^\ell}{\ell} - \frac{s^\ell}{\ell}\right) + O(n^{\ell-1}) \\
&\le \frac{k^\ell n^\ell}{\ell 2^{\ell+1}} + O(n^{\ell-1}),
\end{aligned}$$

where in the first inequality we used Claim 2.18 and pulled the constant out of the sum, and the second inequality follows from $(s+1)^\ell = s^\ell + \ell s^{\ell-1} + O(s^{\ell-2})$. The final inequality follows from the telescoping sum. □

2.3 The number of copies of H in graphs without a certain tree

Alon and Shikhelman, while considering the case when H is a bipartite graph and T is a tree, mention that $\mathrm{ex}(n, H, T) = O(n^{\alpha(H)})$ is a consequence of a theorem from [1]. We prove that, in fact, this holds for general graphs H.

Theorem 2.21 (Győri, Salia, Tompkins, Zamora. [51])**.** *Let H be any graph and let T be any tree, then* $\mathrm{ex}(n, H, T) = O(n^{\alpha(H)})$.

Corollary 2.22. *For any graph H such that $v(H) - \alpha(H) \le \lfloor \frac{k-1}{2} \rfloor$, we have*

$$\mathrm{ex}(n, H, P_k) = \Theta(n^{\alpha(H)}).$$

A construction yielding the lower bound in Corollary 2.22 is $G_{n,k,t}$. Indeed, for every subset of size $\alpha(H)$ of the independent set in $G_{n,k,t}$ we can find a copy of H by joining the t vertices involved in the clique in $G_{n,k,t}$.

Theorem 2.21 follows as a simple consequence of the following lemma which will be proven by induction on $\alpha(H)$.

Lemma 2.23. *For any graph H and any tree T,*

$$\mathrm{ex}(n+1, H, T) - \mathrm{ex}(n, H, T) = O(n^{\alpha(H)-1}).$$

Here, the constant given by the O notation depends only on H and T.

We start by proving the following well-known fact.

Proposition 2.24. *Let H be a graph and let u be a vertex of H. If H' is the graph obtained by removing u together with its neighborhood, then $\alpha(H') \le \alpha(H) - 1$.*

Proof. If X is a maximal independent set in H', then since no neighbor of u is in X, the set $X \cup \{u\}$ is independent in H and so $|X| + 1 = \alpha(H') + 1 \le \alpha(H)$. $\square$

We are now ready to prove Lemma 2.23.

Proof of Lemma 2.23. For the base case of the induction, note that if $\alpha(H) = 1$, then H is a clique and it is simple to see that $\mathrm{ex}(n, K_s, T) = O(n)$ for any s and T. (We may, for example, use the simple bound of $\mathrm{ex}(n, T) \le v(T)n$, for any tree T, and apply an induction argument similar to the proof of Theorem 2.3.)

To estimate $\mathrm{ex}(n+1, H, T) - \mathrm{ex}(n, H, T)$, we will start with a graph G on $n+1$ vertices which is T-free with maximum number of copies of H. We know that $\delta(G) < v(T)$, otherwise $T \subseteq G$. Let v be a vertex of minimum degree in G, and we will count the number of copies of H in G containing v as a vertex. Let $V(H) = \{u_1, u_2, \ldots, u_{v(H)}\}$, and let H_i be the graph obtained by removing u_i together with its neighbors. By Proposition 2.24, we know that $\alpha(H_i) \le \alpha(H) - 1$. Now for each copy of H using v as a vertex, v must play the role of some u_i, and the neighbors of u_i must be embedded in the neighborhood of v. Then the other vertices of H, that is the vertices of H_i, must be embedded in some way in the remaining vertices of G. We have to choose $d_H(u_i)$ vertices in $N(v)$, so the number of copies of H using v is at most

$$\sum_{i=1}^{v(H)} d(v)^{d_H(u_i)} \mathcal{N}(H_i, G) \le \sum_{i=1}^{v(H)} v(T)^{d_H(u_i)} \mathcal{N}(H_i, G) = \sum_{i=1}^{v(H)} O_{H_i}(n^{\alpha(H_i)}) = O(n^{\alpha(H)-1}).$$

Thus, if G' is the graph obtained from G by removing v, we have that

$$\mathrm{ex}(n+1, H, T) = \mathcal{N}(H, G) = \mathcal{N}(H, G') + O(n^{\alpha(H)-1}) \le \mathrm{ex}(n, H, T) + O(n^{\alpha(H)-1}). \quad \square$$

For some particular graphs H, by studying more carefully the number of copies of H that use some fixed vertex, we can find a better recursion than the one from Lemma 2.23. In the following section, we improve the recursion for several specific classes of graphs. For these graphs we will find an integer valued function $f(n)$ which is a lower bound of the extremal number $\mathrm{ex}(n, H, T)$, such that $f(n)$ grows faster than $\mathrm{ex}(n, H, T)$ (when they do not agree). Since both functions are integer valued, they must coincide eventually.

2.4 Exact Results

We now turn our attention to proving some exact results. Recall that we are using the notation $t = \lfloor \frac{k-1}{2} \rfloor$.

2.4.1 Number of copies of C_4

We begin by determining the maximal number of copies of C_4 in a P_k-free graph.

Theorem 2.25 (Győri, Salia, Tompkins, Zamora. [51])**.** *For every integer $k \ge 5$, there exists $n_1 \in \mathbb{N}$ such that if $n \ge n_1$,*

$$\mathrm{ex}(n, C_4, P_k) = \mathcal{N}(C_4, G_{n,k,t}) = \binom{n-t}{2}\binom{t}{2} + 3(n-t)\binom{t}{3} + 3\binom{t}{4} + 2\eta_k\binom{t}{2},$$

where $\eta_k = 1$, if k is even, and 0 otherwise. Moreover, the only extremal graph is $G_{n,k,t}$.

Remark 2.26. *We note however that when $k = 5$, the graph $G_{n,k,t}$ is obtained from $K_{2,n-2}$ by adding an edge in the 2-vertex class, the number of copies of C_4 in $K_{2,n-2}$ is the same as $G_{n,5,2}$. For simplicity a graph G will be considered extremal for $\mathrm{ex}(n, F, H)$ if in addition to maximizing the number of copies of F, the graph G is H-saturated, i.e. it is not possible to add another edge to G without creating a copy of H, since adding edges does not reduce the number of copies of F we can always find such a G.*

To prove Theorem 2.25, we will prove the following claim from which the theorem follows by induction on n.

Claim 2.27. *There exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then*

$$\mathrm{ex}(n+1, C_4, P_k) - \mathrm{ex}(n, C_4, P_k) \leq \binom{t}{2}(n-2).$$

Equality can hold only if the unique extremal graph with $n + 1$ vertices is $G_{n+1,k,t}$.

It is easy to see that $\mathcal{N}(C_4, G_{n+1,k,t}) = \mathcal{N}(C_4, G_{n,k,t}) + \binom{t}{2}(n-2)$. By Claim 2.27, $\mathrm{ex}(n+1, C_4, P_k) \leq \mathrm{ex}(n, C_4, P_k) + \binom{t}{2}(n-2)$ with equality only if the unique extremal graph with $n + 1$ vertices is $G_{n+1,k,t}$. It follows that

$$\mathrm{ex}(n+1, C_4, P_k) - \mathcal{N}(C_4, G_{n+1,k,t}) \leq \mathrm{ex}(n, C_4, P_k) - \mathcal{N}(C_4, G_{n,k,t}),$$

and so the sequence $\mathrm{ex}(n, C_4, P_k) - \mathcal{N}(C_4, G_{n,k,t})$ is a non-increasing sequence of non-negative integers that is strictly decreasing after every non-zero term. Thus, this sequence is eventually the constant 0 sequence, which implies that $G_{n,k,t}$ is eventually the unique extremal graph.

We now prove Claim 2.27.

Proof. Let G be a P_k-free graph on $n + 1$ vertices with the maximum number of copies of C_4, that is, $\mathcal{N}(C_4, G) = \mathrm{ex}(n + 1, C_4, P_k)$.

If $\delta(G) > t$, then by Lemma 2.20, every connected component of G must have size at most k, and therefore $N(C_4, G) \leq 3\binom{k}{4}\frac{n+1}{k} = \frac{(n+1)(k-1)(k-2)(k-3)}{8}$. Then we can choose n_0 so that this number is less than $\mathcal{N}(C_4, G_{n,k,t})$ for $n \geq n_0$, and we would be done.

Thus, we can assume $\delta(G) \leq t$. And suppose $t \geq 2$. Let v be a vertex of minimum degree. By removing v, we are removing at most $\binom{d(v)}{2}(n-2) \leq \binom{t}{2}(n-2)$ copies of C_4. Equality can hold only if $d(v) = t$ and if the neighbors of v have full degree. It follows that if equality holds, then G contains a complete bipartite graph with color classes of size t and $n + 1 - t$ respectively such that the size t class is a clique. If k is odd, we have that $G = G_{n+1,k,t}$. If k is even, since G contains the maximum number of C_4's, it follows that G has an additional edge (it cannot have 2 more for otherwise we would have a P_k). Thus, if k is even we also have $G = G_{n+1,k,t}$. For $t = 2$ we have that for equality to hold $K_{2,n-1} \subseteq G$, and then by maximality we may assume $G = G_{n+1,k,t}$.

Therefore, either $G = G_{n+1,k,t}$ or by removing a minimum degree vertex v we obtain a graph G' with $\mathcal{N}(C_4, G') > \mathcal{N}(C_4, G) - \binom{t}{2}(n-2) = \mathrm{ex}(n+1, C_4, P_k) - \binom{t}{2}(n-2)$. Since $\mathrm{ex}(n, C_4, P_k) \geq \mathcal{N}(C_4, G')$, we have that $\mathrm{ex}(n+1, C_4, P_k) - \mathrm{ex}(n, C_4, P_k) < \binom{t}{2}(n-2)$. $\square$

The same argument proves the following.

Theorem 2.28 (Győri, Salia, Tompkins, Zamora. [51]). *For every positive integer $k \geq 5$, there exists $n_1 \in \mathbb{N}$ such that if $n \geq n_1$*

$$\mathrm{ex}(n, C_4, C_{\geq k}) = \mathcal{N}(C_4, G_{n,k,t}) = \binom{n-t}{2}\binom{t}{2} + 3(n-t)\binom{t}{3} + 3\binom{t}{4} + 2\eta_k\binom{t}{2},$$

where $\eta_k = 1$, if k is even, and 0 otherwise. Moreover, the only extremal graph is $G_{n,k,t}$.

2.4.2 Number of copies of S_r

We will prove the following theorem about the number of copies of P_2. However, it will follow as a consequence of a more general result about stars.

Theorem 2.29 (Győri, Salia, Tompkins, Zamora. [51]). *For every positive integer $k \geq 3$, there exists $n_1 \in \mathbb{N}$ such that if $n \geq n_1$,*

$$\mathrm{ex}(n, P_2, P_k) = \mathcal{N}(P_2, G_{n,k,t}) = t\binom{n-1}{2} + (n-t)\binom{t}{2} + 2t\eta_k,$$

where $\eta_k = 1$, if k is even, and 0 otherwise. Moreover, the only extremal graph is $G_{n,k,t}$.

More generally we have,

Theorem 2.30 (Győri, Salia, Tompkins, Zamora. [51]). *For every positive integer $k \geq 3$ and $r \geq 2$, there exists $n_1 \in \mathbb{N}$ such that if $n \geq n_1$,*

$$\mathrm{ex}(n, S_r, P_k) = \mathcal{N}(S_r, G_{n,k,t}) = t\binom{n-1}{r} + (n-t)\binom{t}{r} + 2\eta_k\binom{t}{r-1},$$

where $\eta_k = 1$, if k is even, and 0 otherwise. Moreover, the only extremal graph is $G_{n,k,t}$, unless k is even and $t \leq r-2$ in which case the only extremal graphs are $G_{n,k,t}$ and $G_{n,k-1,t}$.

Again, the result follows from a claim about the difference of the values of two consecutive extremal numbers. Let $a_n = \mathcal{N}(S_r, G_{n+1,k,t}) - \mathcal{N}(S_r, G_{n,k,t}) = \binom{t}{r} + t\binom{n-1}{r-1}$.

Claim 2.31. *There exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,*

$$\mathrm{ex}(n+1, S_r, P_k) - \mathrm{ex}(n, S_r, P_k) \leq a_n,$$

and equality can hold only if either $G_{n+1,k,t}$ is the only extremal graph on $n+1$ vertices or k is even, $t \leq r-2$ and the only extremal graphs are $G_{n+1,k,t}$ and $G_{n+1,k-1,t}$.

Proof. For any graph G, by counting over all possible centers of a star, we have that $\mathcal{N}(S_r, G) = \sum_{v \in V(G)} \binom{d(v)}{r}$. Let G be a P_k-free graph with $n+1$ vertices and maximum number of copies of S_r; that is, $\mathcal{N}(S_r, G) = \mathrm{ex}(n+1, S_r, P_k)$. We will consider cases depending on the minimum degree of G.

If $\delta(G) > t$, then every connected component of G must have at most k vertices. So the number of copies of S_r is bounded by $n\binom{k-1}{r}$, then we choose n_0 such that this number is less than $\mathcal{N}(S_r, G_{n,k,t})$ for $n \geq n_0$.

If $\delta(G) \leq t$, then by removing v a vertex of minimum degree, we remove at most

$$\binom{d(v)}{r} + \sum_{u \in N(v)} \binom{d(u)-1}{r-1} \leq \binom{t}{r} + t\binom{n-1}{r-1}$$

copies of S_r. Equality can hold only if $d(v) = t$ and the t neighbors of v have degree n, so G contains a complete bipartite graph with color classes of size t and $n+1-t$ such that class of size t is a clique. Then the characterization of the extremal cases again follows from the maximality of G. $\square$

Remark 2.32. *By checking more carefully the difference between the number of r-stars using v and the number a_n, we can find a bound for n_1 of order $k^{3/2}$.*

Similarly as before, the same method proves the following two results.

Theorem 2.33 (Győri, Salia, Tompkins, Zamora. [51])**.** *For every positive integer $k \geq 5$, there exists $n_1 \in \mathbb{N}$ such that if $n \geq n_1$,*

$$\mathrm{ex}(n, P_2, C_{\geq k}) = \mathcal{N}(P_2, G_{n,k,t}) = t\binom{n-1}{2} + (n-t)\binom{t}{2} + 2t\eta_k,$$

where $\eta_k = 1$, if k is even, and 0 otherwise. Moreover the only extremal graph is $G_{n,k,t}$.

Or more generally,

Theorem 2.34 (Győri, Salia, Tompkins, Zamora. [51])**.** *For every positive integer $k \geq 5$, there exists $n_1 \in \mathbb{N}$ such that if $n \geq n_1$,*

$$\mathrm{ex}(n, S_r, C_{\geq k}) = \mathcal{N}(S_r, G_{n,k,t}) = \mathcal{N}(P_2, G_{n,k,t}) = t\binom{n-1}{r} + (n-t)\binom{t}{r} + 2\eta_k\binom{t}{r-1},$$

where $\eta_k = 1$, if k is even, and 0 otherwise. Moreover the only extremal graph is $G_{n,k,t}$, unless k is even and $t \leq r-2$ in which case the only extremal graphs are $G_{n,k,t}$ and $G_{n,k-1,t}$.

Remark 2.35. *For $k = 3$, Theorem 2.33 also holds. Since G must be a tree and by convexity the number of stars is maximized in a star of n vertices, we have $G_{n,3,1} = K_{1,n-1}$, and this graph has $\binom{n-1}{r}$ stars. For $k = 4$, a star with a perfect matching or almost perfect matching in the neighborhood of the center vertex maximizes the number of copies of P_2, with $\binom{n-1}{2} + (n-1)$, when n is odd or $\binom{n-1}{2} + (n-2)$, when n is even. Any graph containing the n vertex star maximizes the number of copies of S_r for $r \geq 3$.*

2.4.3 Number of copies of P_3

Theorem 2.36 (Győri, Salia, Tompkins, Zamora. [51])**.** *For every positive integer $k \geq 5$, there exists $n_1 \in \mathbb{N}$ such that if $n \geq n_1$*

$$\mathrm{ex}(n, P_3, P_k) = \mathcal{N}(P_3, G_{n,k,t}) = \frac{3t(t-1)}{2}n^2 + O(n).$$

Moreover the only extremal graph is $G_{n,k,t}$.

Let a_n be define as

$$\mathcal{N}(P_3, G_{n+1,k,t}) - \mathcal{N}(P_3, G_{n,k,t}) = 2t\left(\binom{t-1}{2} + (n-t)(t-1) + \eta_k\right) + t(t-1)(n-2).$$

As in the previous results it is enough to prove the following Claim.

Claim 2.37. *There exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,*

$$\mathrm{ex}(n+1, P_3, P_k) - \mathrm{ex}(n, P_3, P_k) \leq a_n,$$

and equality can hold only if $G_{n+1,k,t}$ is the only extremal graph on n vertices.

Proof. Let G be an $(n+1)$-vertex graph with maximum number of copies of P_3. We may assume that $\delta(G) \geq 2$. (If a vertex has degree 1, then it is in at most $2(t(n-t) + \binom{t}{2} + \eta_k)$ copies of P_3.)

If $\delta(G) > t$, then each connected component of G must have size at most k, by Lemma 2.20, and so $\mathcal{N}(P_3, G) \leq 3\binom{k}{3}\frac{n+1}{k}$. In this case, we can choose n_0 such that for $n \geq n_0$, this number is less than $\mathcal{N}(P_3, G_{n,k,t})$.

Thus, we may assume that $\delta(G) \leq t$. Let v be a vertex in G with minimum degree, and consider the copies of P_3 containing v as their second vertex.

We may suppose G is connected and has enough vertices to apply Theorem 2.2. Then the number of copies of P_3 whose second vertex is v is bounded from above by

$$d(v)(d(v)-1)(n-2) - 2(d(v)-2)\left(\binom{d(v)}{2} - e(N(v))\right).$$

Indeed, the first term is the trivial upper bound $2\binom{d(v)}{2}(n-2)$ obtained if every pair of neighbors of v could be extended to path of length 3 in any possible way. The subtraction comes from the fact that each non-edge $\{a, b\}$ in the neighborhood of v along with a third neighbor $c \in N(v)$ uniquely forbids 2 copies of P_3 namely *cvab* and *cvba*. We have bounded from above the number of copies of P_3 containing v as a second vertex.

Now we will obtain an estimate on the number of copies of P_3 starting at v. We consider the number of ways to take distinct $u \in N(v), w \in N(u)$ and $x \in N(w)$:

$$\begin{aligned}
&\sum_{u\in N(v)} \sum_{\substack{w\in N(u)\\ w\neq v}} \left(d(w) - 1 - \mathbb{1}_{w\in N(v)}\right) = \sum_{u\in N(v)} \left(\sum_{\substack{w\in N(u)\\ w\neq v}} d(w) - d(u) + 1\right) - 2e(N(v)) \\
&= \sum_{u\in N(v)} \left(\sum_{w\in V(G)} d(w) - \sum_{\substack{w\notin N(u)\\ w\neq u}} d(w) - d(v) - 2d(u)\right) - 2e(N(v)) + d(v) \\
&= \sum_{u\in N(v)} \left(2e(G) - \sum_{\substack{w\notin N(u)\\ w\neq u}} d(w) - 2d(u)\right) - 2e(N(v)) - d(v)(d(v)-1) \\
&\leq \sum_{u\in N(v)} \Big(2e(G) - 2(n - d(u)) - 2d(u)\Big) - 2e(N(v)) - d(v)(d(v)-1) \\
&= 2d(v)(e(G) - n) - 2e(N(v)) - d(v)(d(v)-1),
\end{aligned}$$

where for the second equality we used that

$$\sum_{\substack{w\in N(u)\\ w\neq v}} d(w) = \sum_{w\in V(G)} d(w) - \sum_{\substack{w\notin N(u)\\ w\neq u}} d(w) - d(u) - d(v),$$

and the inequality uses that $\delta(G) \geq 2$.

The above sum is maximized when $d(v) = t$, and to achieve this maximum it is necessary that for every neighbor u of v, either each non-neighbors of u have degree 2 or that $\{w \notin N(u) : w \neq u\} = \emptyset$.

When $t \geq 3$, we have the bound (conditioning on whether v is at the beginning or middle of the path)

$$2t(e(G)-n) - 2e(N(v)) - t(t-1) + t(t-1)(n-2) - 2(t-2)\left(\binom{t}{2} - e(N(v))\right),$$

$$\leq 2t(e(G)-n) - 2t(t-1) + t(t-1)(n-2).$$

From Theorem 2.2 it follows that this number is at most a_n. To obtain equality, in both cases it is necessary that every neighbor of v has full degree and so by maximality we have that $G = G_{n+1,k,t}$.

If $t = 2$ ($k = 5$ or $k = 6$) we obtain a bound of

$$2(n-3) + 4(e(G)-n) - 2e(N(v)),$$

where is $e(N(v))$ is 0 or 1, it follows that the maximum is only achieve when $e(G) = e(G_{n,k,2}) = 2(n-2) + 1 + \eta_k$, which implies that $G = G_{n+1,k,2}$ so $e(N(v)) = 1$, hence the previous maximum is $6(n-3) - 2 + 4\eta_k = a_n$. □

Remark 2.38. *For $k = 4$, it is simple to check that the only extremal graph is a balanced double star on n vertices, which has $\lfloor \frac{n-1}{2} \rfloor \lceil \frac{n-1}{2} \rceil$ copies of P_3.*

Now we consider paths of length 4.

2.4.4 Number of copies of P_4

Theorem 2.39 (Győri, Salia, Tompkins, Zamora. [51])**.** *For every positive integer $k \geq 5$, there exists $n_1 \in \mathbb{N}$ such that if $n \geq n_1$, then*

$$\mathrm{ex}(n, P_4, P_k) = \mathcal{N}(P_4, G_{n,k,t}) = \frac{t(t-1)}{2} n^3 + \Theta(n^2).$$

Moreover the only extremal graph is $G_{n,k,t}$.

Similarly as before, let $a_n := \mathcal{N}(P_4, G_{n+1,k,t}) - \mathcal{N}(P_4, G_{n,k,t})$. We have that

$$a_n = 2t\mathcal{N}(P_2, G_{n-1,k-2,t-1}) + 2t(t-1)e(G_{n-2,k-4,t-2}) + \binom{t}{2}(n-2)(n-3).$$

As in the previous results it is enough to prove the following Claim.

Claim 2.40. *There exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,*

$$\mathrm{ex}(n+1, P_4, P_k) - \mathrm{ex}(n, P_4, P_k) \leq a_n,$$

and equality can hold only if $G_{n+1,k,t}$ is the only extremal graph on n vertices.

Proof. Let G be an $(n+1)$-vertex graph with maximum number of copies of P_4. Assume that $\delta(G) \geq 2$. (If a vertex v has degree 1, then v is in at most $2\mathcal{N}(P_2, G_{n,k,t}) < a_n$ copies of P_4.)

If $\delta(G) > t$ then, by Lemma 2.20, each component of G must have size at most k and so $\mathcal{N}(P_4, G) \leq 12\binom{k}{4}\frac{n+1}{k}$ so we can choose n_0 such that if $n \geq n_0$ this number is less than $\mathcal{N}(P_4, G_{n,k,t})$.

Suppose now that $\delta(G) \leq t$. Let v be a vertex of minimum degree. As before suppose G is connected. To count the number of paths of length 4 starting at v, fix $u \in N(v)$ and let G' be the subgraph of G obtained by removing v and u. Any path $vuu_1u_2u_3$ can be decomposed as the edge vu together with the ordered path $u_1u_2u_3$ in G' so the number of paths of the form $vuu_1u_2u_3$ is at most $2\mathcal{N}(P_2, G')$, since there are two orderings of any P_2. It is easy to check that G' cannot contain a cycle of length at least $k-1$, otherwise together with the edge uv we would have a copy of P_k.

Thus, there are two cases:

a) Suppose first that G' is does not contain a C_{k-2}. Then G' is $C_{\geq k-2}$-free and so $\mathcal{N}(P_2, G') \leq ex(n-1, P_2, C_{\geq k}) = \mathcal{N}(G_{n-1,k-2,t-1}, P_2)$. We will take n_0 bigger than the constant from Theorem 2.33, when $k = 5$, or $k \geq 7$. When $k = 6$ we use the following lemma.

Lemma 2.41. *If H is a graph on m vertices containing no cycle of length at least 4, then either H contains a vertex of degree $m-1$ or $\mathcal{N}(P_2, H) < \binom{m-1}{2} + 2$.*

Proof. Suppose H has no vertex of degree $m-1$. If H has degree 1 vertices, then the number of copies of P_2 is maximized when all these vertices are adjacent to the vertex of maximum degree, so suppose H has no vertex of degree 1. By Theorem 1.25

$$\sum_{v \in V(H)} d(v) \leq 3(m-1) \text{ and } 2 \leq d(v) \leq m-2.$$

Hence, by convexity, the number of copies of P_2, $\displaystyle\sum_{v \in H} \binom{d_H(v)}{2}$, is maximized when there is one vertex of degree $m-2$, one of degree 3 and $m-2$ of degree 2. This yields $\binom{m-2}{2} + m + 1 = \binom{m-1}{2} + 3$ paths of lenght 2, however a graph with such a degree sequence must have a cycle of length four. Thus, we consider the second best degree sequences, which has one vertex of degree $m-2$ and $m-1$ vertex of degree 2 (if possible) which has $\binom{m-1}{2} + 1$ copies of P_2. $\square$

Now according to this lemma for $k = 6$, either $\mathcal{N}(P_2, G') < \binom{n-2}{2} + 2 = \mathcal{N}(P_2, G_{n-1,4,1})$ or G' has a vertex w of degree $n-2$, and some number s of independent edges in $N_{G'}(w)$. The vertex u cannot be adjacent to two different edges in $N_{G'}(w)$, otherwise G would contain a P_6, so u is adjacent to at most $n+1-s$ vertices of G', this number is achieved if u is adjacent to w and if u is adjacent to both vertices of one of these edges, then the number of copies of P_4 starting with vu would be $(n-2s)(n-3)+2+2s \leq (n-2)(n-1)+4$.

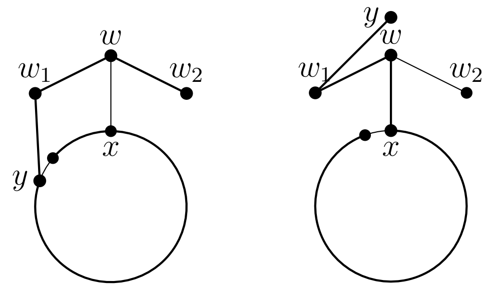


Figure 2.5: Cycle in proof of Theorem 2.39.

b) Now suppose that G' contains a cycle of length $k-2$, C. In this case we have the following.

Claim 2.42. *If w is a vertex which is not in the cycle and $w \in N(x)$ where x is a vertex of the cycle, then w has at most one neighbor outside of C.*

Proof. Suppose w_1 and w_2 are two neighbors of w. Since $\delta(G) > 1$, w_1 has a neighbor y. If y is in C, then C together with w_1ww_2 is a length k path. If y is outside of C, then C together with ww_1y is a P_k. □

As a corollary we have,

Claim 2.43. *If w is a vertex not in the cycle and $w \in N(x)$ where x is a vertex of the cycle, then $d(w) < k$.*

The edge uv is connected to C by some path. If there is an intermediate vertex from C to uv, then clearly this path will have length at least k and so this is not possible. Hence C is connected to either u or v. If C is connected to v, then every neighbor of u must be in C for otherwise we have a k path. If C is connected to u, then by Claim 2.42, all neighbors of u, except for v, are in C. So for every path $vuu_1u_2u_3$ we have that $u_1 \in C$, if $u_2 \in C$, then we have less than k choices for both u_1 and u_2 and at most n choices for u_3. If u_2 is not in C, then by Claim 2.43, since u_2 is a neighbor of $u_1 \in V(C)$, we have $d(u_2) < k$ and so there are at most k choices for u_3 and less than n choices for u_2. Hence we have less than $2k^2n$ such paths in total and we can take n_0 such that if $n \geq n_0$, then this number is less than $2\mathcal{N}(G_{n-1,k-2,t-1}, P_2)$.

It follows that the number of paths starting with v is at most $2d(v)\mathcal{N}(P_2, G_{n-1,k-2,t-1})$.

Now if $d(v) \leq t-1$, then the trivial bound on the number of copies of P_4 with middle vertex v is $d(v)(d(v)-1)\binom{n-2}{2}$ and the bound on the number of P_4 cpoies with v as a second vertex is $2d(v)(d(v)-1)e(G)$. Thus, we would have that the number of copies of P_4 using v is less than a_n. So we will now suppose $d(v) = t$. To simplify the notation in the following calculations let $S := \sum_{u \in N(v)} d(u)$.

To count paths with v in the middle, we will count in order paths of the form $xuvwy$, where u, w can be any neighbors of v and then we have to choose a neighbor x of u and a neighbor y of w with $y \neq x$. Hence the number of ordered paths with v as the middle vertex is

$$\sum_{u \in N(v)} \left(\sum_{\substack{w \in N(v) \\ w \neq u}} (d(u) - 1 - \mathbb{1}_{u \in N(w)})(d(w) - 1 - \mathbb{1}_{u \in N(w)}) - |N(u) \cap N(w)| + 1 \right)$$

$$
\begin{aligned}
&\le \sum_{u\in N(v)}\left(\sum_{\substack{w\in N(v)\\ w\neq u}} d(u)d(w)-2d(w)-2d(u)-\mathbb{1}_{u\in N(w)}(d(w)+d(u))+5\cdot\mathbb{1}_{u\in N(w)}+n+1\right)\\
&= S^2-\left(\sum_{u\in N(v)} d(u)^2\right)-4(t-1)S-2\left(\sum_{u\in N(v)} d(u)\,|N(u)\cap N(v)|\right)+10e(N(v))+\\
&\quad t(t-1)(n+1)\\
&\le S^2-\left(\sum_{u\in N(v)} d(u)^2\right)-4(t-1)S-2\left(\sum_{u\in N(v)} d(u)(d(u)+t-n-1)\right)+t(t-1)(n+6)\\
&= S^2-3\left(\sum_{u\in N(v)} d(u)^2\right)+(2n-6(t-1))S+t(t-1)(n+6)\\
&\le S^2-\frac{3S^2}{t}+(2n-6(t-1))S+t(t-1)(n+6)\\
&= \frac{t-3}{t}S^2+(2n-6(t-1))S+t(t-1)(n+6),
\end{aligned}
$$

where in the first and second inequality we use the fact that for every pair of vertices x, y of the graph $|N(x)\cap N(y)|\geq d(x)+d(y)-n+1-2\cdot\mathbb{1}_{x\in N(y)}$ together with $e(N(v))\leq\binom{t}{2}$. The last inequality was obtained by applying the Cauchy-Schwarz inequality to $\sum_{u\in N(v)} d(u)^2$. Since any path can have two distinct orders, we divide this expression by 2.

To count the number of paths with v as the second vertex, we will decompose the path $uvwxy$ into uvw together with $e = xy$. First we choose in order two neighbors of v, then an edge not using u, v or w. There are at most 2 ways to connect the edge to w, so the number of these paths is at most

$$
\begin{aligned}
&2\sum_{u\in N(v)}\left(\sum_{\substack{w\in N(u)\\ w\neq u}} e(G)-d(w)-d(u)-t+2+\mathbb{1}_{u\in N(w)}\right)\\
&\qquad= 2t(t-1)(e(G)-t+2)+4e(N(v))-4(t-1)S\\
&\qquad\le 2t(t-1)(e(G_{n,k,t})+2)+4\binom{t}{2}-4(t-1)S.
\end{aligned}
$$

By summing the previous bounds, we have that the number of paths using v is at most

$$
2t\mathcal{N}(P_2, G_{n,k-2,t-1})+\frac{t-3}{2t}S^2+(n-7(t-1))S+t(t-1)(n+6)+2t(t-1)(e(G_{n,k,t})+2)+4\binom{t}{2}.
$$

The value of this expression when $S = tn$ is precisely a_n. By considering this expression as a quadratic in S, we can check that if $t\geq 3$ the maximum is attained only when $S = nt$. This means that every neighbor of v must have degree n, so this is only possible if $G = G_{n+1,k,t}$. If $t = 2$, the expression attains its maximum when $S = 2n-14$, hence if $S < 2n-28$. This value would be less than a_n, but now with the condition $S\geq 2n-28$. It is simple to check that for $k = 5$, G must be either $K_{2,n-2}$ or $G_{n,5,2}$, and if $k = 6$, then G must be $G_{n,6,2}$. □

2.5 The number of copies of P_{k-1} in P_k-free graphs

If k is odd, it seems likely that the graph $G_{n,k,t}$ attains the value $\mathrm{ex}(n, P_{k-1}, P_k)$. However, for k even the situation changes. We have that $\mathcal{N}(P_{k-1}, G_{n,k,t}) = \Theta(n^t)$, but there is another graph $H_{n,k}$ such that $\mathcal{N}(P_{k-1}, H_{n,k}) = \Theta(n^{t+1})$. In order to define this graph, first for $r \geq 2$ and a, b positive integers, let $S_{a,b}^{(r)}$ be the $(a+b+r)$-vertex graph consisting of a clique on r vertices and two independent sets A and B on a and b vertices, respectively, then for a fixed vertex v be a vertex of the clique, join v to every vertex in B, then join every vertex of the clique except v to every vertex in A. Let $\mathcal{S}_n^{(r)}$ be the family of all such graphs on n vertices. For even k, let $H_{n,k}$ be the graph in $\mathcal{S}_n^{(t+1)}$ which maximizes the number of P_{k-1} copies. In this case we conjecture that the graph $H_{n,k}$ is extremal.

Conjecture 2.44. *If k is even and $k \geq 4$, the extremal number $\mathrm{ex}(n, P_{k-1}, P_k)$ is attained by the the graph $H_{n,k}$.*

Remark 2.45. *For $r \geq 2$ the graphs $S_{a,b}^{(r)}$ are P_{2r}-free and have*

$$\mathcal{N}(P_{2r-1}, S_{a,b}^{(r)}) = (r-1)!\, ba(a-1)\cdots(a-r+2).$$

In $\mathcal{S}_n^{(r)}$ this number is maximized when a is roughly $\frac{r-1}{r}n$, and this maximum approaches $(r-1)!\,(\frac{(r-1)^{r-1}}{r^r})n^r$ as n tends to infinity. In particular by taking $r = 3$ we have a P_6-free graph with $8n^3/27 + O(n^2)$ copies of P_5.

Remark 2.46. *Note that the only edges of the clique in $S_{a,b}^{(r)}$ that a P_{2r-1} uses are the ones that are incident with the vertex v. So, we have several graphs for which we conjecture the number of copies of P_{2r-1} is maximal, namely those subgraphs of $S_{a,b}^{(r)}$ formed by removing edges from the clique not incident to v.*

Conjecture 2.44 can be easily checked for $k = 4$, and the following theorem says that this conjecture is also true for $k = 6$.

Theorem 2.47 (Győri, Salia, Tompkins, Zamora. [51])**.** *There exists $n_1 \in \mathbb{N}$ such that if $n \geq n_1$, then*

$$\mathrm{ex}(n, P_5, P_6) = \mathcal{N}(P_5, H_{n,6}) = \frac{8n^3}{27} + O(n^2).$$

Proof. Let G be a P_6 free graph and suppose $n \geq 7$. It is enough to bound the copies of P_5 in each connected component, so assume G is connected.

Let C be the largest cycle in G and let G' be the graph obtained by removing C from G; clearly C cannot be a 6-cycle, otherwise G would contain a P_6. We will consider cases based on the length of C.

a) Suppose C is a 5-cycle with vertices v_1, v_2, v_3, v_4, v_5 appearing consecutively. Then every vertex in G' is connected to a vertex of C. Suppose that v_1 has a neighbor in G', if v_1 is the only vertex of C with an edge to G', then $\mathcal{N}(P_5, G) < 24n$. So, suppose this is not the case, v_2 and v_5 cannot have neighbors in G'. Thus, without loss of generality, we may suppose v_3 has a neighbor in G', then v_4 cannot have neighbors in G', also note that v_2 cannot be connected with v_4 or with v_5 and so $G \subseteq G_{n,6,2}$ (where the v_1 and v_3 take the role of the high degree vertices, and the edge v_4v_5 is the only edge that is not incident with one of v_1 or v_3), hence $\mathcal{N}(P_5, G) = O(n^2)$.

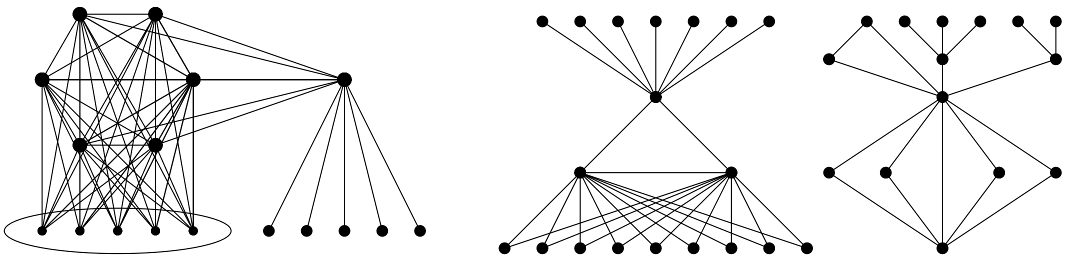


Figure 2.6: The family $\mathcal{S}_{a,b}^{(r)}$ from which the conjectured extremal graph $H_{n,k}$ is obtained. And a graph that appears in Case b) of the proof of the Theorem 2.47.

b) Now suppose C is a 4-cycle defined by v_1, v_2, v_3, v_4, consecutively. Then G' cannot contain a P_3, otherwise by connectivity we would have a path of length at least 6. Consider the set X of vertices of G' that have at least one neighbor in both C and G'. Note that if $y \in G'$ is a neighbor of $x \in X$, then y cannot have any other neighbor in G'. Also note that the only possible neighbor of y in C is the neighbor of x in C (y cannot have a neighbor in C if x has more than one neighbor in C).

If $|X| > 1$, then every vertex in X must be adjacent to the same vertex in C, say v_1. Then v_2 and v_4 cannot have a neighbor outside of C. If v_2 and v_4 are adjacent, then it also holds that v_3 cannot have neighbors in G'. It is then easy to check that $\mathcal{N}(P_5, G) < 6n$. So, suppose v_2 and v_4 are not connected (see Figure 2.6), then every P_5 in G is of the form xyv_1vv_3u, where $x \in X$, y is a neighbor of x, and both of v, u are common neighbors of v_1 and v_3. If $a = |N(v_1) \cap N(u) \cap N(v)|$ and b is the number vertices in G' with a neighbor in X, then we have that $\mathcal{N}(P_5, G) \leq ba(a-1)$ which is half $\mathcal{N}(P_5, S_{a,b}^{(3)})$ but $S_{a,b}^{(3)}$ can have at most one more vertex than G.

If $X = \{x\}$, then something similar holds, except that both v_1 and v_3 are allow to be connected to x and there is the extra possibility of G being a subgraph of $S_{a,b}^{(3)}$.

Now suppose $X = \emptyset$. If no two vertices of C share a common vertex in G', then $\mathcal{N}(P_5, G)$ is quadratic. So, suppose two non-consecutive vertices, say v_1, and v_3, share a common neighbor, then it is not possible for the other two vertices in C to have a neighbor in G'. Thus, our graph is again a subgraph of $S_{a,b}^{(3)}$.

c) Suppose C is a triangle, then every pair of vertices are the end vertices of at most one P_5. If two different paths of length 5 have the same end vertices, then either G would contain a cycle of length at least four or a P_6. Thus, $\mathcal{N}(P_5, G) < \binom{n}{2}$. $\square$

Chapter 3

The Maximum number of Pentagons in Planar Graphs

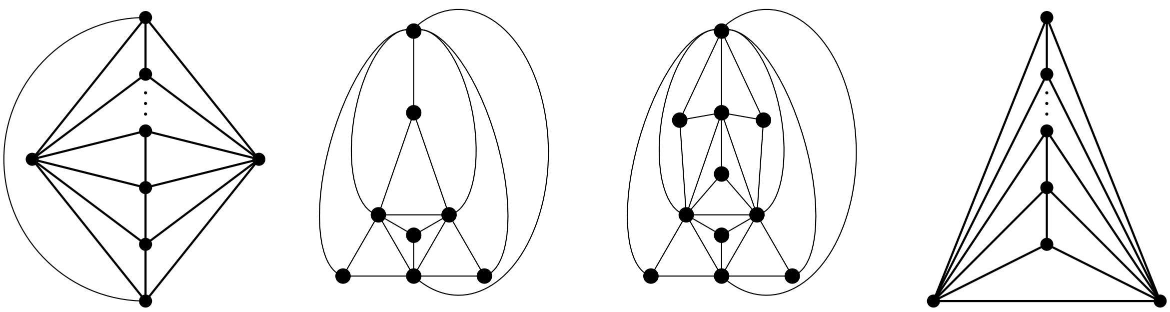
3.1 Introduction

Let $f(n, H) := \mathrm{ex}_p(n, H, \emptyset)$ denote the maximum number of copies of the graph H in an n-vertex planar graph.

Hakimi and Schmeichel [53] proved that $f(n, C_5) \leq 5n^2 - 26n$. Furthermore, they conjectured that a bound of $2n^2 - 10n + 12$ should hold, which is attained by the graph D_n pictured in Figure 3.1. We confirm that their conjecture holds (for $n \geq 8$), and provide a complete characterization of the extremal graphs for all n. Our main result is the following.

Theorem 3.1 (Győri, Paulos, Salia, Tompkins, Zamora. [50])**.** *For $n = 6$ and $n \geq 8$, $f(n, C_5) = 2n^2 - 10n + 12$. We have $f(5, C_5) = 6$ and $f(7, C_5) = 41$. Moreover the planar graphs that maximize the number of copies of C_5 are the family of graphs D_n obtained from a cycle on $n - 2$ vertices by adding two vertices that are adjacent to each vertex of the cycle (see Figure 3.1). When $n = 8$ or $n = 11$ the graphs A_n (see Figure 3.1) also achieve the maximum.*

Recall that a copy of H in a graph G is a subgraph of G (not necessarily induced), isomorphic to H. For graphs G and H, we denote by $\mathcal{N}(H, G)$ the number of copies of H



(a) D_n (b) A_8 (c) A_{11} (d) E_n

Figure 3.1: The graphs D_n, A_8, A_{11} and E_n.

in G. The neighborhood of a vertex v is denoted by $N(v)$, and the closed neighborhood (that is, $\{v\} \cup N(v)$) is denoted by $N[v]$. If $C = x_1x_2 \dots x_kx_1$ be a cycle in G, then C is said to separate vertices $y, z \in V(G)$ in a planar embedding of G if one of y or z is in the interior of the curve formed by embedding of the cycle and the other one is in the exterior. For a given graph G, if $e = \{v, u\}$ is an edge of G, then the contraction of the edge e is the graph obtained from G by replacing the two vertices $\{v, u\}$ with a new vertex w and replacing the edges of the form $\{v, x\}$ and $\{y, u\}$ with the edges $\{w, x\}$ and $\{y, w\}$, taking the new edges without multiplicity.

Thought this chapter, a path of length k traversing the vertices $x_1, x_2, \dots, x_{k+1}$ in that order is denoted $x_1x_2 \dots x_{k+1}$. Similarly, a cycle of length k going through the vertices $x_1, x_2, \dots, x_k, x_1$ is denoted by $x_1x_2 \dots x_kx_1$.

3.2 Proof of Theorem 3.1

We start with the proof of three lemmas which will be used later in the proof of Theorem 3.1.

Lemma 3.2. *Let G be a planar graph and $\{u, v\}$ be an edge, then $G[N(u) \cap N(v)]$ is a path forest. Moreover the graph induced by $\{u, v\} \cup (N(u) \cap N(v))$ is a triangulation if and only if $G[N(u) \cap N(v)]$ is a path.*

Proof. First, we show that $G' = G[N(u) \cap N(v)]$ is acyclic. Suppose by contradiction, there is a cycle C in the common neighborhood $N(u) \cap N(v)$, then we have a K_5-minor. Indeed, if we contract this cycle to a triangle, then this triangle together with v and u forms a K_5, contradicting planarity. Hence G' is acyclic.

Now we show that $d_{G'}(w) \leq 2$ for all $w \in V(G')$. Suppose $d_{G'}(w) \geq 3$ for some $w \in V(G')$, then taking three distinct vertices $w_1, w_2, w_3 \in N(u) \cap N(v) \cap N(w)$ yields a $K_{3,3}$, a contradiction to planarity. Therefore, G' is a path forest.

The graph induced by the vertex set $\{v, u\} \cup (N(u) \cap N(v))$ is a triangulation if and only if it has $3\,|N(u) \cap N(v)|$ edges. There are $2\,|N(u) \cap N(v)| + 1$ edges incident with u or v. So, the graph induced by $\{v, u\} \cup (N(u) \cap N(v))$ is a triangulation if and only if we have the graph induced by $N(u) \cap N(v)$ has precisely $|N(u) \cap N(v)| - 1$ edges. Therefore, it is a triangulation if and only if $G[N(u) \cap N(v)]$ is a path. □

Lemma 3.3. *Let G be a planar graph on $k \geq 3$ vertices and let $\{u, v\}$ be an edge of G, then the number of length three paths from u to v is at most $2(k - 3)$.*

Proof. We may assume that $N(u) \cap N(v) \neq V(G) \setminus \{u, v\}$, since otherwise Lemma 3.2 would imply the result. Indeed, since $G[N(u) \cap N(v)]$ is a path forest, it has at most $k - 3$ edges and so there are at most $2(k - 3)$ paths of length 3 between u and v.

We are going to prove the lemma by induction on k, the result is trivial for $k = 3$. Consider the set of ordered pairs $A = \{(x, y) : uxyv \text{ is a path}\}$. Let x be a vertex that is not in $N(u) \cap N(v)$, and without loss of generality suppose $x \notin N(v)$. If x is in at most two pairs from A, then we can remove x and we would be done by induction, so suppose x is in at least three pairs from A. It follows that there exist three vertices distinct from u, say y_1, y_2 and y_3, in $N(x) \cap N(v)$. Suppose further that y_2 is such that the cycle xy_1vy_3x separates y_2 and u (see Figure 3.2). Then y_2 is not adjacent to u. Now contract

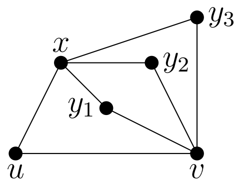


Figure 3.2: Separating cycle from proof of Lemma 3.3.

the edge $\{x, y_2\}$ to a vertex x', and note that with the exception of uxy_2v, every path of length 3 from u to v using either x or y_2 yields a unique path of length 3 from u to v containing x' after the contraction. So, by induction, in the contracted graph we have at most $2(k-4)$ paths of length 3 from u to v, therefore in the original graph we have at most $2(k-4)+1 < 2(k-3)$ such paths. $\square$

Lemma 3.4. *Let $k \geq 4$, G be a k-vertex planar graph and let T be the set of three vertices of any triangular face of G, then the number of paths of length three with both end vertices in T is at most $4(k-1)$. If there is no vertex adjacent to all vertices in T, then there are at most $4k-9$ such paths.*

Proof. First, we will prove the lemma holds in the case where every vertex of G is adjacent to at least two vertices of T. Let x_1, x_2, x_3 be the vertices of T, and let $A = N(x_1) \cap N(x_2) \setminus \{x_3\}$, $B = N(x_2) \cap N(x_3) \setminus \{x_1\}$ and $C = N(x_3) \cap N(x_1) \setminus \{x_2\}$. Note that there is at most one vertex in the intersection $N(x_1) \cap N(x_2) \cap N(x_3)$.

Suppose that there exists a vertex u incident to every vertex of T. The vertex u can have at most one neighbor in A and similarly at most one neighbor in B and C. The number of paths of length 3 from x_1 to x_2 using x_3 is precisely $|B|+|C|$, since each such path is of the form $x_1x_3bx_2$ with $b \in B$ or $x_1cx_3x_2$ with $c \in C$. There are at most 2 paths of length 3 from x_1 to x_2 using u and a vertex not in A, since u has at most one neighbor in B and at most one neighbor in C (the only vertices in $N(x_1) \cup N(x_2) \cup N(x_3)$ which are not in $A \cup B \cup C$ are the vertices of T). All other paths of length 3 from x_1 to x_2 have internal vertices only from A, so by Lemma 3.3, we have at most $2|A|-2$ such paths. Thus, we have a bound of $2|A|+|B|+|C|$ on the number of length 3 paths from x_1 to x_2 and similarly a bound of $|A|+2|B|+|C|$ for the number of 4-vertex external paths from x_2 to x_3 and $|A|+|B|+2|C|$ from x_3 to x_1. Thus, in total we have at most $4(|A|+|B|+|C|) = 4(k-1)$ paths of length 3 between vertices of T.

If there is no vertex adjacent with the three vertices of T, then we have at most one edge between A and B, and the same for B and C as well as C and A. In a similar way, we get a bound of $4(|A|+|B|+|C|)+3 = 4k-9$ on the number of paths of length 3 between vertices of T.

Now, we are ready to prove the general case. Suppose there is a vertex x which is adjacent to at most one vertex of T. If x is in at most four paths of length 3 between vertices of T, then we can remove x and the result would follow by induction (induction on k of the statement of the lemma; the base case $k=4$ is trivial). If x is in at least five such paths, then x must have a neighbor in T, say x_1, hence these paths have the form x_1xyx_i for some y and $i=2$ or $i=3$. Without loss of generality, there are three paths of the form $x_1xy_1x_2$, $x_1xy_2x_2$ and $x_1xy_3x_2$. Thus, we have than one of y_1, y_2 or y_3, say y_2,

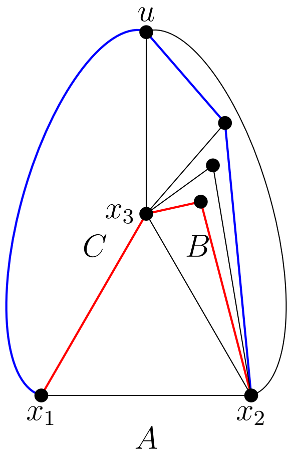


Figure 3.3: Two possible length 3 paths from x_1 to x_2.

cannot be adjacent to x_1 or x_3. Therefore, if we contract the edge $\{x, y_2\}$ to a vertex z, the number of paths of length 3 between vertices of T increases by one. The only path that is lost is $x_1xy_2x_2$, while the two new paths $x_1zx_2x_3$ and $x_3x_1zx_2$ appear. Thus, we are done by induction. $\square$

Remark 3.5. *If one of the sets A, B or C is empty, then we get a bound of $4(k-1)-2$ on the number of length 3 paths with both end vertices in T. This holds since if one of these sets is empty, say $A = \emptyset$, then by adding a vertex w in the face $x_1x_2xux_1$, adjacent to the three vertices of that face, we would create 6 such paths. Applying Lemma 3.4 to the resulting $(k+1)$-vertex graph, we would obtain a bound of $4k$ on the number of paths of length 3 between vertices of T, and so we have a bound of $4(k-1)-2$ on the number of such length 3 paths in the original graph. With a similar argument it follows that if only one of A, B, C is nonempty, then we obtain a bound of $4(k-2)$.*

Remark 3.6. *Since G is triangulated, the neighborhood of every vertex has a Hamiltonian cycle.*

Proof of Theorem 3.1. It is simple to check that $f(5, C_5) = 6$, since there is precisely one maximal planar graph on 5 vertices. Let $g(n) = 2n^2 - 10n + 12$ for $n \neq 7$, and $g(7) = 2 \cdot 7^2 - 10 \cdot 7 + 13 = 41$. The lower bound, $f(n, C_5) \geq g(n)$, for $n \geq 6$ can by checked by taking the graph D_n, which has $g(n)$ copies of C_5. For the upper bound, let G be a maximal planar graph that maximizes the number of copies of C_5. We may suppose G is a triangulated planar graph on n vertices and that G is embedded in the plane. The proof proceeds by induction on the statement that $f(n, C_5) \leq g(n)$. For the base case $n = 5$, we are already done, since $f(5, C_5) = 6 \leq 12 = g(5)$.

For any $(n-1)$-vertex graph G' we have

$$f(n, C_5) \leq f(n-1, C_5) + \mathcal{N}(C_5, G) - \mathcal{N}(C_5, G').$$

Note that for $n \neq 8, 7$, $g(n) - g(n-1) = 4(n-3)$, while $g(7) - g(6) = 17$ and $g(8) - g(7) = 19$. Therefore, by defining an $(n-1)$-vertex graph G', such that the difference $\mathcal{N}(C_5, G) - \mathcal{N}(C_5, G')$ is bounded by $4(n-3)$, the proof of the upper bound would follow by induction (except when $n = 8$, where we need a bound of $4(n-3) - 1$).

We will divide the proof into several cases according to whether there is a vertex of degree 4 or 3 or the minimum degree is 5. For simplicity, we will assume that the first vertex v which we fix is in the interior of the cycle defined by its neighborhood.

Case 1. Suppose G has a vertex of degree 4, say v. In this case, let $\{v_1, v_2, v_3, v_4\}$ be the neighborhood of v such that $v_1v_2v_3v_4v_1$ is a cycle. Note that it is not possible for both of the edges $\{v_1, v_3\}$ and $\{v_2, v_4\}$ to be present in G, since otherwise we would have a K_5.

Without loss of generality, suppose the edge $\{v_2, v_4\}$ is not present, and consider the graph G' obtained by removing v and adding the edge $\{v_2, v_4\}$. We are going to bound $\mathcal{N}(C_5, G) - \mathcal{N}(C_5, G')$. Equivalently, we bound the number of copies of C_5 in G containing v minus the number of copies of C_5 in G' that use the edge $\{v_2, v_4\}$.

We say that a 5-cycle in G containing v is of *type* k, if it contains precisely k vertices which are not in $N[v]$, and we say that a 5-cycle in G' is of *type* k, if it contains the edge $\{v_2, v_4\}$ and precisely k vertices which are not in $N(v)$. Let $\mathcal{N}_k$ be the number of type k cycles in G minus the number of cycles of type k in G' ($\mathcal{N}_3 \leq 0$, while $\mathcal{N}_k = 0$ for $k \notin \{0, 1, 2, 3\}$). Note that

$$\mathcal{N}(C_5, G) - \mathcal{N}(C_5, G') = \mathcal{N}_0 + \mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3 \leq \mathcal{N}_0 + \mathcal{N}_1 + \mathcal{N}_2.$$

We say that a path is *internal* if all of its vertices are in $N[v]$. We say a path is *external* if all of its non-terminal vertices are not in $N[v]$. (Note that a path can be neither external nor internal.)

We will count the number of type 1 cycles using a vertex x according to whether they use consecutive or nonconsecutive vertices in the 4-cycle.

Note that $\mathcal{N}_0$ is just the number of 5-cycles in the graph induced by $N[x]$, thus $\mathcal{N}_0$ is either 4 or 6 depending on whether $\{v_1, v_3\}$ is present.

For each $x \in (N(v_i) \cap N(v_{i+1})) \setminus N[v]$ (where indices are taken modulo 4) and $i = 1, 2, 3, 4$, the cycles of type 1 in G using the path v_ixv_{i+1} are $vv_ixv_{i+1}v_{i+2}v$ and $vv_{i-1}v_ixv_{i+1}v$. If $\{v_1, v_3\}$ is an edge, then we have one of $vv_ixv_{i+1}v_{i-1}v$ or $vv_{i+2}v_ixv_{i+1}v$ (according to whether i is odd or even), but in this case we also have one of the following type 1 cycles in G': $v_{i+2}v_ixv_{i+1}v_{i-1}v_{i+2}$ or $v_{i-1}v_{i+2}v_ixv_{i+1}v_{i-1}$ (see Figure 3.4).

For each $x \in (N(v_i) \cap N(v_{i+2})) \setminus N[v]$, we have only the following four cycles of type 1 in G: $vv_ixv_{i+2}v_{i-1}v$, $vv_ixv_{i+2}v_{i+1}v$, $vv_{i-1}v_ixv_{i+2}v$, $vv_{i+1}v_ixv_{i+2}v$ (note that in this case $\{v_{i-1}, v_{i+1}\}$ is not an edge). If $i = 1$ (or $i = 3$), we have in G' the type 1 cycles $v_2v_1xv_3v_4v_2$ and $v_4v_1xv_3v_2v_4$ (see Figure 3.4). For $i = 2$ (or $i = 4$), we have no type 1 cycles in G' using v_2xv_4. Therefore, we have

$$\mathcal{N}_1 = \sum_{1 \leq i < j \leq 4} h(i, j) \left| N(v_i) \cap N(v_j) \setminus N[v] \right|, \tag{3.1}$$

where $h(i, j)$ is the number of internal 4-vertex paths from v_i to v_j containing v, that is, $h(2, 4) = 4$ and $h(i, j) = 2$ for every other pair (i, j), $1 \leq i < j \leq 4$.

Note that for every type 2 cycle in G of the form $vv_ixyv_{i+1}v$, we have one following type 2 cycles in G': $v_2v_ixyv_{i+1}v_2$ for $i = 3, 4$ or $v_4v_ixyv_{i+1}v_4$ for $i = 1, 2$ (see Figure 3.4). Thus, the number of type 2 cycles in G is bounded by the number of 4-vertex external paths from v_1 to v_3 and from v_2 to v_4.

Case 1. a) Suppose the edge $\{v_1, v_3\}$ is present. In this case there is no path joining v_2 and v_4 without using v, v_1 or v_3. It follows that the number of type 2 cycles in G is at most the number of 4-vertex external paths from v_1 to v_3. Note that there is at most one vertex in $V(G) \setminus N[v]$ which is adjacent to v_1, v_2 and v_3 as well as at most one adjacent to v_3, v_4 and v_1. Every other vertex is adjacent to at most two vertices of $N(v)$.

For $n = 6$, we must have that G is E_6, which has 20 five cycles, so suppose $n \geq 7$. We will assume that there exists a vertex u, $u \neq v$, adjacent to v_1, v_2, v_3 and a vertex w, $w \neq v, u$, adjacent to v_3, v_4, v_1, since the result follows in a similar way in the other cases. We have that $\mathcal{N}_0 = 6$ and $\mathcal{N}_1 \leq 12 + 2(n-7)$ by (3.1). Indeed, this is true since both u and w each contribute to six type 1 cycles and every other vertex (outside of $N[v]$) contributes at most two to $\mathcal{N}_1$. Equality holds if and only if every vertex other than u is in $V \setminus N[v]$, and w is adjacent to precisely two vertices of $N(v)$.

By applying Lemma 3.3 to the regions determined by the triangles $v_1v_2v_3v_1$ and $v_3v_4v_1v_3$ (containing u and w, respectively), we have at most $2(n-7)$ 4-vertex external paths from v_1 to v_3, and there are at least two type 2 cycles in G', namely $uv_2v_4wv_3u$ and $uv_2v_4wv_1u$, hence $\mathcal{N}_2 \leq 2(n-7) - 2$. Thus, we have

$$\mathcal{N}(C_5, G) - \mathcal{N}(C_5, G') = \mathcal{N}_0 + \mathcal{N}_1 + \mathcal{N}_2 \leq 4(n-3).$$

Note that if every vertex in $V \setminus N[v]$ besides u and w is adjacent to precisely two vertices of $N(v)$, then there are precisely $n-7$ edges not incident with $N(v)$. Since every 4-vertex external path from v_1 to v_3 is of the form v_1xyv_3, where the edge $\{x, y\}$ is not incident to $N(v)$, we have a bound of $2(n-7)$ on the number of such paths, and equality is only possible if every such vertex in $V \setminus N(v)$ is adjacent to both v_1 and v_3. Therefore, either G is E_n, which has $2n^2 - 10n + 8$ copies of C_5, or $\mathcal{N}(C_5, G) - \mathcal{N}(C_5, G') < 4(n-3)$. For $n \neq 8$ we are done, while for $n = 8$ it is simple to check that G' cannot be D_7, so G' can have at most 40 copies of C_5 and we are also done by induction.

Case 1. b) Suppose the edge $\{v_1, v_3\}$ is not present and that there does not exist another vertex distinct from v which is adjacent to v_1, v_2, v_3 and v_4. Note that in $n \geq 7$ and we have that $\mathcal{N}_1 = 4$.

In this case there cannot exist two distinct vertices $x_1, x_2 \in V(G)$ with $x_1, x_2 \neq v$ such that $x_1 \in N(v_1) \cap N(v_3)$ and $x_2 \in N(v_2) \cap N(v_4)$. Thus, without loss of generality, we may assume $N(v_2) \cap N(v_4) = \{v\}$. As in the previous case, there is at most one vertex $u, u \neq v$, adjacent to v_1, v_2, v_3, and there is at most one vertex $w, w \neq v$, adjacent to v_3, v_4, v_1. We will assume that both vertices exist, since the other cases are similar.

Let X be the set of vertices in $V \setminus N(v)$ which are adjacent to precisely two vertices of $N(v)$, and let Y be the set of vertices which are adjacent to just one vertex of $N(v)$.

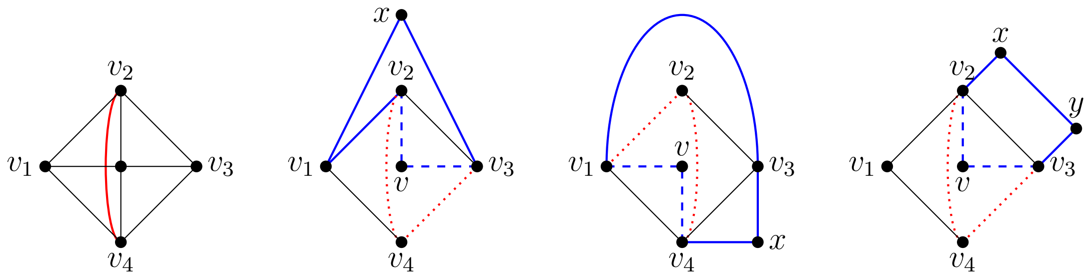


Figure 3.4: The figure shows several cycles in G containing v which can be transformed into cycles of G' with the additional edge $\{v_2, v_4\}$; the new cycles are obtained by replacing the two dashed lines by the two dotted lines.

Both u and w contribute to six type 1 cycles of $\mathcal{N}_1$ and every vertex of X contributes to two. By equation (3.1) we have that $\mathcal{N}_1 \leq 12 + 2|X|$.

Suppose that w and u are adjacent, then we have the 4-vertex external path v_2uwv_4 from v_2 to v_4, and two 4-vertex external paths from v_1 to v_3 using the vertices u and w: v_1uwv_3 and v_1wuv_3. For each vertex $z \in V \setminus N[v]$, besides u and w, any 4-vertex external path from v_1 to v_3 must include u or w, but if z is in two such paths, then z must be adjacent to both u and w. We would then have the type 3 cycle in G' given by xuv_2v_4wx, while z cannot be in any 4-vertex external path from v_2 to v_4, hence z contributes to at most one to $\mathcal{N}_2 + \mathcal{N}_3$. Note that for every $x \in X$, since x must be adjacent to two consecutive vertices of $N(v)$, we have a type 2 cycle in G'. For instance, if x is adjacent to v_1 and v_2, we have the 5-cycle $xv_2v_4wv_1x$. Since we have the paths v_1uwv_3, v_2uwv_4 and v_3uwv_1 and the type 2 cycles $uv_2v_4wv_3u$ and $uv_2v_4wv_1u$ in G', we have that

$$\mathcal{N}_2 + \mathcal{N}_3 \leq 1 + (n-7) - |X|.$$

Therefore $\mathcal{N}(C_5, G) - \mathcal{N}(C_5, G') \leq 17 + (n-7) + |X| \leq 17 + 2(n-7)$, which is less than $4(n-3)$ if $n \geq 8$. If $n = 7$, in fact G is equal to D_7, and this graph has 41 copies of C_5. For $n = 8$, we have that $\mathcal{N}(C_5, G) - \mathcal{N}(C_5, G') \leq 19$, however it is easy to check that the graph G' is not D_7, so G' must have at most 40 five cycles, and it follows that G could have at most 59 five cycles.

If u and w are not adjacent (note that this is only possible if $n \geq 8$), then there is no 4-vertex external path from v_2 to v_4. Then by Lemma 3.3, we have at most $2(n-6)$ 4-vertex external paths from v_1 to v_3. Since we have at least two additional type 2 cycles in G', namely $uv_2v_4wv_3u$ and $uv_2v_4wv_1u$, we have that $\mathcal{N}_2 \leq 2(n-6) - 2$. Therefore

$$\mathcal{N}(C_5, G) - \mathcal{N}(C_5, G') \leq 14 + 2(n-6) + 2|X| \leq 4(n-3),$$

which for $n \geq 9$ implies by induction that $\mathcal{N}(C_5, G) \leq 2n^2 - 10n + 12$. Moreover, equality is only possible if $|X| = n-7$, that is, if every vertex in $V \setminus N[v]$ except for u and w is adjacent to precisely two vertices of $N(v)$. In this case, there are $n-6$ edges not incident with $N(v)$, so the number of 4-vertex external paths from v_1 to v_3 is at most $2(n-6)$, with equality if and only if each vertex in $V \setminus N(v)$ is adjacent to both v_1 and v_3, and so equality can only be achieved if G is D_n. When $n = 8$, since G is a maximal planar graph, we have that there must be a vertex adjacent to v_1, v_3, u and w, and so G must be D_8.

Case 1. c) Suppose the edge $\{v_1, v_3\}$ is not present and there exists a vertex u, $u \neq v$, which is adjacent to v_1, v_2, v_3 and v_4. Observe that for $n = 6$, G is equal to D_6 which has 24 five cycles, so suppose $n \geq 7$. Note that there cannot be any vertex other than u in $V \setminus N[v]$ which is adjacent to both v_1 and v_3 or both v_2 and v_4. Let X be the set of vertices that are adjacent to precisely two vertices of $N(v)$ and let Y be the set of vertices which are adjacent to just one vertex of $N(v)$, by equation (3.1) we have $\mathcal{N}_1 = 14 + 2|X|$.

Observe that every vertex of Y is in at most one 4-vertex external path from v_1 to v_3 or one 4-vertex external path from v_2 to v_4, and each of these paths must go through the vertex u. A vertex $x \in X$ can only be in two such paths but this only occurs if x is adjacent to u. Since u can be adjacent to at most one vertex of X in the region bounded by $uv_iv_{i+1}u$, there are at most $2\min\{4, |X|\} \leq 8$ such 4-vertex external paths using u and a vertex of X. Also note that every $x \in X$ is in a type 2 cycle of G' (for example, if x is

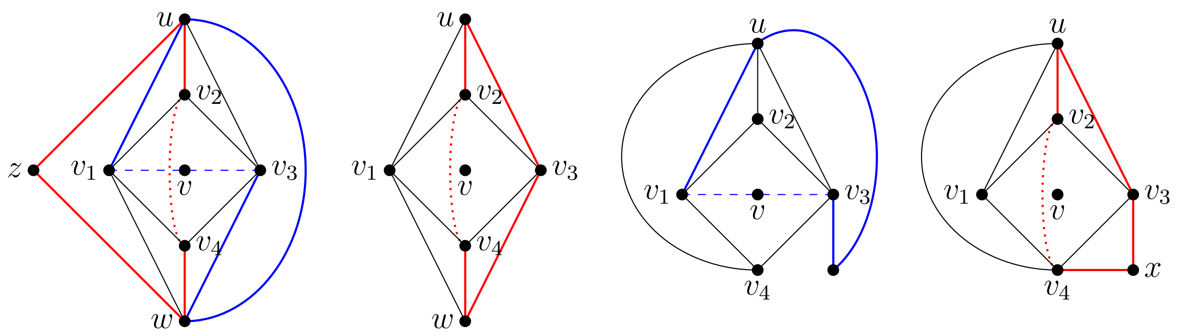


Figure 3.5: The first pictures show a type 3 cycle in G' in Case 1.b when u, v have a common neighbor and a type 2 cycle in G when the edge $\{u, w\}$ is present. The second picture shows one of the two type 2 cycles in G' that uses both u and w. The last two pictures show a type 2 cycle in Case 1.c. There is one such cycle for every common neighbor of u and v_i not in $N(v)$. The last picture shows a type 2 cycle from Case 1.c that occurs in G' for each $x \in X$, without using the edge $\{u, x\}$.

adjacent to v_3 and v_4, then $xv_3uv_2v_4x$ is a five cycle in G'). Therefore $\mathcal{N}_2 \leq |Y| - |X| + 8$, and it follows that

$$\mathcal{N}(C_5, G) - \mathcal{N}(C_5, G') \leq 18 + |X| + |Y| + 8 \leq 18 + (n-6) + 8.$$

This last bound is strictly less than $4(n-3)$ if $n \geq 11$.

Before dealing with the smaller cases ($n = 7, 8, 9, 10$) we note the following: if there is precisely one vertex x inside the region $v_iv_{i+1}u$, then this vertex x is adjacent to the three vertices v_i, v_{i+1}, u and no other vertex can be adjacent to these three vertices. We can apply induction in this case by removing x, and the details will be handled in the next case (Case 2.a). If there are precisely two vertices x, y in the region $uv_iv_{i+1}u$, then one of these two vertices, say x, will have degree 4 and have the neighborhood $\{v_i, v_{i+1}, y, u\}$. By taking the vertex x, we would be in Case 1.a.

For the smaller values of n we have the following. If $n = 7$, clearly there will be precisely one vertex in one of the four regions $uv_iv_{i+1}u$. If $n = 8$, then either we have two vertices in the same region (Case 1.a) or two regions with one vertex (Case 2.a, which is considered later). When $n = 9$ or $n = 10$, if the other four vertices of G are in the same region $v_iv_{i+1}u$, then $\mathcal{N}_2 \leq |Y| - |X| + 2$. It follows that

$$\mathcal{N}(C_5, G) - \mathcal{N}(C_5, G') \leq 20 + |X| + |Y| < 4(n-3),$$

so, either we are done by induction or there are at least two regions containing vertices inside. In the latter case, by the pigeonhole principle, we must find a region with either 1 or 2 vertices, and so we end up in Case 1.a or Case 2.a.

Case 2. Suppose G has a vertex v of degree 3, and let $\{v_1, v_2, v_3\}$ be its neighborhood. Then $v_1v_2v_3v_1$ is a triangle, and we will assume v is in the interior of the triangle. Let G' be the graph induced by $G[V \setminus v]$. The number of five cycles in G containing v is precisely the number of 4-vertex paths with end vertices in $N(v)$, the vertices of $N(v)$ are the vertices of a triangular face in G' so we will be able to use Lemma 3.4 in the different cases.

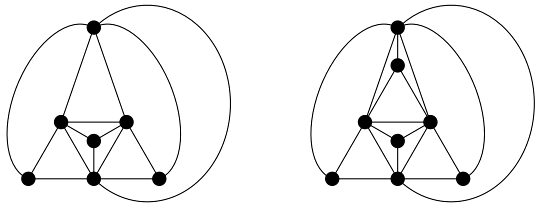


Figure 3.6: Exceptional graphs with 7 and 8 vertices and 36 and 60 five cycles respectively. The graph A_8 is an extremal example.

We will distinguish two cases based on whether or not there exists another vertex adjacent to the three vertices of $N(v)$.

Case 2. a) Suppose there is no vertex u, $u \neq v$, which is adjacent v_1, v_2 and v_3. By Lemma 3.4 there are at most $4(n-3)-1$ five cycles containing v. Thus, by induction, if $n \geq 6$, $n \neq 8$, we have $\mathcal{N}(C_5, G) < 2n^2 - 10n + 12$. For $n = 8$, this gives a bound of at most 60 five cycles, which is only possible if G is obtain by D_7 by adding a vertex in a triangular face, it is simple to check that these graphs have 57 five cycles.

Case 2. b) Suppose there is a vertex u, $u \neq v$, which is adjacent to the three vertices of $N(v)$. We can also assume that there is no vertex of degree 4, since otherwise we would be done by the previous cases. If two of the regions uv_iv_ju, $i, j \in \{1, 2, 3\}$, say uv_1v_2u and uv_2v_3u, are empty. It follows that the vertex v_2 would be a 4-degree vertex. So, we have that at most one of these regions is empty. We note that in particular, for $n = 6$, there is no such graph, while for $n = 7$ and $n = 8$, there is just one such graph which has 36 and 60 five cycles, respectively (see Figure 3.6).

Suppose $n \geq 9$. It is simple to check that G' cannot be D_{n-1}, A_8 or A_{11}. Indeed, D_{n-1} has $n-2 \geq 7$ vertices of degree 4 and since not all of them are adjacent to v, then G would already have a vertex of degree 4. Both A_8 and A_{11} have one vertex of degree 3 in each of their faces, but one of these vertices would be adjacent to v and there would be a vertex of degree 4 in G. We have then that $\mathcal{N}(C_5, G') \leq (n-1)^2 - 10(n-1) + 11$ and so if v is in at most $4(n-1)$ five cycles, we would be done by induction.

If there is at least one vertex which is not adjacent to any vertex of $N(v)$, then such a vertex does not appear in any 4-vertex path joining vertices of $N(v)$. Thus, by Lemma 3.4, v would be in at most $4(n-3)$ five cycles and we would be done by induction.

Thus, we may assume that every vertex is adjacent to at least one vertex of $N(v)$. Let

$$X = \{x \in V(G) \setminus N(v) : |N(x) \cap N(v)| = 2\}$$

and

$$Y = \{y \in V(G) : |N(y) \cap N(v)| = 1\}.$$

Note that every vertex of Y can be adjacent to at most 2 vertices of $X \cup \{u\}$, since otherwise we would have a a copy of $K_{3,3}$ in G. We can conclude that each vertex of Y is in at most three 4-vertex paths with both end vertices in $N(v)$, and this if can be achieved only if y is adjacent to both u and a vertex of X, otherwise y is in at most 2 such cycles. If we remove the vertices of Y and apply Lemma 3.4, we obtain a bound of $4(n-3) + 4 - |Y|$ on the number of these 4-vertex paths. So, we can suppose that $|Y| \leq 3$, otherwise we would be done by induction. Additionally, if $y \in Y$ is such that $N(y) \cap N(v) = \{v_i\}$, we can suppose that every neighbor of y is also a neighbor of v_i.

Indeed, if there was a vertex w which is a neighbor of y but not of v_i, then by contracting the edge $\{y, w\}$, the number of 4-vertex paths with both end vertices in $N(v)$ would increase by one (as in the proof of Lemma 3.4). Then, by applying Lemma 3.4 in the contracted graph, we would get a upper bound of $4(n-3)-1$ on the number of 4-vertex paths with both end vertices in $N(v)$. In particular, this implies the following claim.

Claim 3.7. *For each $i \in \{1, 2, 3\}$, the set of vertices of X in the region bounded by the triangle $uv_iv_{i+1}u$ together with u induces a path.*

Proof. Let X' be the set of vertices of X inside the triangle $uv_iv_{i+1}u$, and let $m = |X'|$. If $m = 0$ there is nothing to prove, so suppose $m \geq 1$ and let $x_1, x_2, \ldots, x_m$ be the vertices of X', ordered such that for each $s = 2, 3, \ldots, m$, the vertices $x_1, x_2, \ldots, x_{s-1}$ are inside the region $x_sv_iv_{i+1}x_s$. For simplicity let $x_{m+1} = u$, we are going to show that x_s is adjacent to x_{s+1} and so, $x_1, x_2, \ldots, x_m$ is a path. Since $N(v_s)$ is a cycle, it follows that, besides the edge v_iv_{i+1}, there is another path from v_i to v_{i+1} containing only vertices of $N(v_s)$, this path only uses vertices in the close region $x_sv_ix_{s+1}v_{i+1}x_s$. Then this path must contain the vertex x_{s+1}, for otherwise there would be a path $v_iy_1y_2 \ldots y_rv_{i+1}$ such that $y_1, \ldots, y_r$ are all in Y. However, there must be a first index k such that y_k is adjacent to v_i and y_{k+1} is incident with v_{i+1}, which is impossible. □

We will distinguish two further subcases based on whether at least one of the regions of $uv_iv_{i+1}u$, $i = 1, 2, 3$ is empty.

Case 2. b)* Suppose one of these regions is empty, say uv_1v_2u. By Remark 3.5, v is in at most $4(n-3)+2-|Y|$ five cycles. So if $|Y| \geq 2$, then v would be in at most $4(n-3)$ five cycles, and we would be done by induction. Thus, we can suppose $|Y| \leq 1$, and moreover if $|Y| = 1$, then the vertex of Y must be adjacent to u, since otherwise it would be in at most two 4-vertex paths with both end vertices in $N(v)$.

By Claim 3.7, it follows that X together with u induces a path P on $|X|+1$ vertices. Hence, the path P has at least $|X|-2$ internal vertices excluding u. Note that these internal vertices have exactly four neighbors in $V \setminus Y$.

For $n = 9$, if $Y = \emptyset$, then P has at least two internal vertices, and so a vertex of degree 4. If $|Y| = 1$, then P has one internal vertex x distinct from u. If the vertex of Y is not adjacent to x, then we have a vertex of degree 4, and so we are done. If the vertex of Y is adjacent to x, then there are two possible graphs, and we can check that the number of 5-cycles in the resulting graphs is 79 and 80 (see Figure 3.7).

If $n \geq 10$, then $|X| \geq 4$. Hence there are at least two internal vertices in P distinct from u, and if $Y \neq \emptyset$, the single vertex of Y can be adjacent to at most one of these vertices and so have degree 4. Therefore, we are done by Case 1.

Case 2. b)** Suppose now that for each i, j there is at least one vertex in the region uv_iv_ju, so the graph A_8 is contained in G.

If $n = 9$, then wherever the last vertex is added we will have a vertex of degree 4 and so we would be done by Case 1. If $n = 10$ or $n = 11$, then there is precisely one such graph with no vertex of degree 4 (see Figure 3.7), and these graphs have, respectively, 110 and 144 five cycles. We note that for $n = 11$ this graph is A_{11} the other extremal construction, since $2 \cdot 11^2 - 10 \cdot 11 + 12 = 144$.

If $n \geq 12$, then $|X| \geq 4$. In the interior of one of the regions, say uv_1v_2u, we are able to find at least two vertices of X. Let x_1 be a vertex in this region, such that $x_1v_1v_2x_1$

is a face and let x_2 be its neighbor from X. If x_1 is adjacent to precisely one vertex of Y, then x_1 would have degree 4. If x_1 is adjacent with two vertices of Y, then these two vertices are not adjacent to u and so both are in at most 2 five cycles containing v. Therefore, v would be in at most $4(n-3)$ five cycles.

So, suppose x_1 has degree 3, with neighborhood $\{v_1, v_2, x_2\}$. If there is no vertex that is adjacent to both v_1 and x_2 but not adjacent to v_2, then we would be back in Case 2.b)*. So, suppose such a vertex y_1 exists. Note that y_1 must be in Y, since x_2 is inside the region uv_1v_2u, and every vertex of X in this region is adjacent to both v_1 and v_2. Analogously, we can suppose there is a vertex $y_2 \in Y$ adjacent to x_2 and v_2. Now if these vertices are not adjacent to u, then v would be in at most $4(n-3)$ five cycles and we would be done by induction. If these two vertices are adjacent to u, then there would be precisely two vertices of X in the region uv_1v_2u. By the same reasoning, if there is another region uv_iv_ju with at least two vertices of X, then we would be able to find at least two more vertices of Y or we would be in Case 2.b)*. However, the former possibility is cannot occur since $|Y| \leq 3$. If there is precisely one vertex of X in the regions uv_2v_3u and uv_3v_1u, then $|X| = 4$, and so $n \leq 12$. Hence $n = 12$, $|Y| = 3$ and the vertex of X in one of the regions uv_2v_3u or uv_3v_1u has degree 4, therefore we are done by case 1.

Case 3. Suppose there is no vertex of degree 3 or 4, then the minimum degree of G is 5. Let v be a vertex of degree 5, and let $v_1, v_2, \ldots, v_5$ be the neighbors of v arranged such that $v_1v_2v_3v_4v_5v_1$ is a cycle. There must exists some i such that both edges $\{v_i, v_{i-2}\}$ and $\{v_i, v_{i+2}\}$ are missing (where indices are taken modulo 5). Without loss of generality, let v_1 be such a vertex. Also, it is not possible for both edges $\{v_2, v_4\}$ and $\{v_3, v_5\}$ to be present in G, so we will assume $\{v_3, v_5\}$ is not an edge. Consider the graph G' obtained from deleting v and adding the edges $\{v_1, v_3\}$ and $\{v_1, v_5\}$.

As in Case 1, we say that a 5-cycle in G containing v is *type* k, for $k = 0, 1, 2$, if it contains v and precisely k vertices not in $N[v]$, and we say that a five cycle in G' is *type* k, for $k = 0, 1, 2$, if it contains at least one of the edges $\{v_1, v_3\}$ or $\{v_1, v_4\}$ and precisely k vertices not in $N(v)$. Let $\mathcal{N}_k$ be the number of type k cycles in G minus the number of type k cycles in G'. We say that a path is *internal* if all of it vertices are in $N[v]$. We say a path is *external* if all of its vertices which are not endpoints are not in $N[v]$.

As in Case 1, the number of type 1 cycles is

$$\sum_{1 \leq i < j \leq 5} h(i,j) \left| (N(v_i) \cap N(v_j) \setminus N[v] \right|, \tag{3.2}$$

where $h(i,j)$ is the number of internal 4-vertex paths from v_i to v_j containing v minus the number of internal 4-vertex paths from v_i to v_j using at least one of the edges $\{v_1, v_3\}$

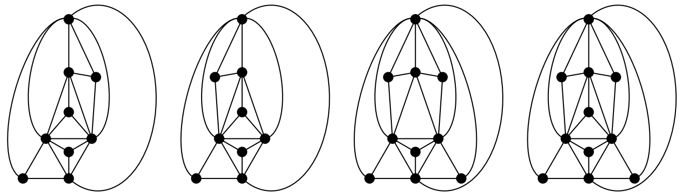


Figure 3.7: Exceptional graphs with 9,9,10 and 11 vertices and 79,80,110 and 144 five cycles respectively.

or $\{v_1, v_4\}$ (since for each such path and $x \in N(v_i) \cap N(v_j) \setminus N[V]$, we have a type 1 cycle). Without the possible edges $\{v_2, v_5\}$ and $\{v_2, v_4\}$ for $i, i+1$, we have the internal 2 internal paths containing v, $v_i v v_{i+2} v_{i+1}$ and $v_i v_{i-1} v v_{i+1}$. While for i and $i+2$ we have the paths $vv_{i-1}v_i x v_{i+2} v$, $vv_{i+1}v_i x v_{i+2} v$, $vv_i x v_{i+2} v_{i+1} v$ and $vv_i x v_{i+2} v_{i+3} v$.

In G' we have the following internal 4-vertex paths using at least one of the edges $\{v_1, v_3\}$ or $\{v_1, v_4\}$: $v_1v_4v_3v_2$, $v_1v_3v_4v_5$, $v_3v_1v_5v_4$, $v_3v_2v_1v_4$, $v_2v_1v_4v_3$, $v_5v_1v_3v_4$, $v_2v_1v_3v_4$, $v_2v_3v_1v_4$, $v_5v_1v_4v_3$, $v_5v_4v_1v_3$, $v_2v_1v_4v_5$, $v_2v_3v_1v_5$, $v_1v_3v_4v_5$.

We can also check that for every internal 4-vertex path containing v, and at least one of the edges $\{v_2, v_5\}$ or $\{v_2, v_4\}$ (in the case they are present in G), except for the path $v_4v_2vv_5$, there is a corresponding internal 4-vertex path in G' (see Figure 3.8).

We have the following chart:

$\{i,j\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{1,5\}$	$\{2,3\}$	$\{2,4\}$	$\{2,5\}$	$\{3,4\}$	$\{3,5\}$	$\{4,5\}$
$h(i,j)$	1	4	4	1	1	2	2	0	2	1 *or* 2

Where $h(4,5) = 1$ if $\{v_2, v_4\}$ is not an edge of G, and $h(4,5) = 2$ otherwise.

Note that for any 4-vertex path $v_i xyv_j$ with $x, y \notin N[v]$, there is precisely one five cycle containing the path and v, namely $v_i xyv_j vv_i$. While in G' we have that for $\{i,j\} \neq \{2,5\}$ there is one 5-cycle using at least one of the edges $\{v_1, v_3\}$ or $\{v_1, v_4\}$. The cycles are the following: If $1 \notin \{i,j\}$ the 5-cycle $v_i xyv_j v_1 v_i$ uses at least one of the edges $\{v_1, v_3\}$ or $\{v_1, v_4\}$ (see Figure 3.8). If $i = 1$ and $j \in \{2,4\}$ the 5-cycle $v_1 xyv_j v_3 v_1$ uses the edge $\{v_3, v_1\}$. Finally, if $i = 1$ and $j \in \{3,5\}$ the 5-cycle $v_1 xyv_j v_4 v_1$ uses the edge $\{v_1, v_4\}$.

Observe that there are 5, 10 or 17 type 0 cycles in G if zero, one or two of the edges $\{v_2, v_5\}$ or $\{v_2, v_4\}$ are present, respectively. On the other hand, in G' the number of type 0 cycles is zero if none of these edges is present, while there is one type 0 cycle using only $\{v_4, v_2\}$, three using only $\{v_2, v_5\}$ and one using both. Hence $\mathcal{N}_0 \in \{5, 7, 9, 12\}$, accordingly.

We will make use of the following fact. If G is a planar graph with minimum degree 5 and $abca$ is a triangular region with one vertex in its interior, then this region must contain at least 3 vertices.

Case 3. a) Suppose there is a vertex u, $u \neq v$, which is adjacent to every vertex of $N(v)$. Note that in this case, neither of the edges $\{v_2, v_5\}$ or $\{v_2, v_4\}$ can be present in G, so $\mathcal{N}_0 = 5$. We have that u contributes to 18 type 1 cycles, and every other vertex is adjacent to at most two vertices of $N(v)$, which must be consecutive. In particular any other vertex contributes to at most two type 1 cycles, so $\mathcal{N}_1 \leq 18 + 2(n-7)$. Finally, we have that each of these vertices can be in at most one 4-vertex path from v_2 to v_5, since each such path must include u, so $\mathcal{N}_2 \leq (n-7)$. Hence $\mathcal{N}(C_5, G) - \mathcal{N}(C_5, G') \leq 23 + 3(n-7)$. We have that for any two consecutive regions, $uv_{i-1}v_i u$, $uv_i v_{i+1} u$, one must contain a vertex in its interior, since otherwise v_i would have degree four. So at least three of the regions $uv_i v_{i+1} u$, must contain a vertex in their interior. However since the minimum degree is 5, it follows that each such region must have at least 3 vertices, therefore $n \geq 17$ and for these values of n we have that $23 + 3(n-7) < 4(n-3)$.

Case 3. b) Suppose there exists a vertex $u \in V$ that is adjacent to precisely four vertices of $N(v)$. Without loss of generality, suppose these 4 vertices are v_4, v_5, v_1 and v_2. We have that u contributes to at most 12 type 1 cycles and Since v_2 and v_5 cannot be adjacent, we have $\mathcal{N}_0 \leq 9$.

As in the previous case every vertex in $V \setminus N[v]$ different from u can be in at most one 4-vertex path from v_2 to v_5, so $\mathcal{N}_2 \leq (n-7)$. Note that it is not possible for all three of the regions uv_4v_5u, uv_5v_1u and uv_1v_2u to be empty since each of v_1 and v_5 must still have another neighbor, and so one of these faces must have at least 3 vertices.

If there exists a vertex $w \in V$ that is adjacent to v_2, v_3 and v_4, then we also have that one of the regions wv_4v_3w or wv_3v_2w must contain at least 3 vertices hence $n \geq 14$. The vertex w contributes to 3 type 1 cycle, and every other vertex contributes to at most 2 type 1 cycles. Thus, in this case we have $\mathcal{N}_1 < 12 + 3 + 2(n-8)$, hence

$$\mathcal{N}(C_5, G) - \mathcal{N}(C_5, G') < 22 + 3(n-7) \leq 4(n-3).$$

If no vertex is adjacent to v_2, v_3 and v_4, then the vertices $x \in N(v_3) \setminus N[v]$ contribute to at most 1 type cycle, since v_3 is not adjacent to v_1 or v_3 and $d(v) \geq 5$, there are at least to such vertices, we have that $\mathcal{N}_1 \leq 12 + 2 + 2(n-9)$, so

$$\mathcal{N}(C_5, G) - \mathcal{N}(C_5, G') \leq 19 + 3(n-7) < 4(n-3),$$

since $n \geq 12$.

Case 3. c) Suppose that every vertex in $V \setminus N[v]$ is adjacent to at most 3 vertices of $N(v)$. Note also that if there is an external path of length 2 from two non consecutive vertices of $N(v)$, say v_i, x, v_{i+2}, then there cannot exists an external path of length 2 from v_{i+1} to either v_{i-1} or v_{i+3}. So we can assume, without loss of generality, that v_1 is such that there is no external path of length two from v_1 to v_3 or v_4.

If $\{v_2, v_4\}$ is an edge of G, then we have that no vertex inside the region $v_2v_3v_4v_2$ can be in an external 4-vertex path from v_2 to v_5, and since this region must contain at least 3 vertices, we have by Lemma 3.3 that $\mathcal{N}_2 \leq 2(n-10)$.

There can exist at most one vertex distinct from v which is adjacent to v_2, v_4 and v_5. Such vertex would contribute to 6 type 1 cycles. Also, there is at most one vertex distinct to v adjacent to v_3, v_4, v_2. Such vertex would contribute to 3 type 1 cycles. Every other vertex contributes to at most 2 type 1 cycles. Therefore $\mathcal{N}_1 \leq 9 + 2(n-8)$, since $\mathcal{N}_0 \leq 12$ we have

$$\mathcal{N}(C_5, G) - \mathcal{N}(C_5, G') \leq 25 + 4(n-10) < 4(n-3).$$

If $\{v_2, v_4\}$ is not an edge of G, then $\mathcal{N}_0 \leq 7$. In this case, there can exist at most one vertex that is adjacent to one of v_2 or v_5 and one of v_3 or v_4, such vertex would

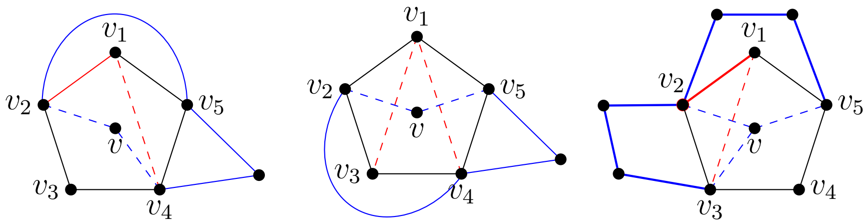


Figure 3.8: In the first picture we have a type 1 cycle G using the edge $\{v_2, v_5\}$ and its corresponding cycle in G'. In the second picture we have a type 1 cycle with a vertex in $N(v_4) \cap N(v_5)$ using the edge $\{v_2, v_4\}$; this cycle has no corresponding cycle in G'. The last picture shows a pair of type 2 cycles, only those using v_2 and v_5 do not have a corresponding type 2 cycle in G'.

contribute to 5 type 1 cycles. And there can exists at most one vertex that is adjacent to both v_3, v_4 and one of v_2 or v_5, such vertex would contribute to 3 type 1 cycles. Therefore $\mathcal{N}_1 \leq 8 + 2(n-8)$. Finally, by Lemma 3.3, we have $\mathcal{N}_2 \leq 2(n-7)$, thus

$$\mathcal{N}(C_5, G) - \mathcal{N}(C_5, G') \leq 13 + 4(n-7) < 4(n-3). \qquad \square$$

Chapter 4

Turán numbers of Berge trees

4.1 Background

We begin by recalling Erdős and Gallai Theorem on P_k-free graphs (Theorem 1.24).

Theorem (Erdős, Gallai [23]). *Let n, k be positive integers and let G be an n-vertex graph containing no path of k edges, then*

$$e(G) \leq \frac{(k-1)n}{2}.$$

Equality is obtained if and only if k divides n and G is the graph consisting of n/k disjoint complete graphs of size k.

Erdős and Sós [22] conjectured that the same bound would hold for any graph not containing a copy of some tree with k edges. A proof of this conjecture for sufficiently large k was announced in the 90's by Ajtai, Komlós, Simonovits and Szemerédi. We will consider a variant of this problem in the setting of hypergraphs and multi-hypergraphs.

We recall first the definitions of Berge graphs and the Turán number of a family of hypergraphs.

Definition. *Given a graph G, a hypergraph $\mathcal{H}$ is a* Berge copy *of G, if there exists an injection $f_1 : V(G) \to V(\mathcal{H})$ and a bijection $f_2 : E(G) \to E(\mathcal{H})$, such that if $e = \{v_1, v_2\} \in E(G)$, then $\{f_1(v_1), f_1(v_2)\} \subseteq f_2(e)$.*

Definition. *The Turán number of a family of r-uniform hypergraphs $\mathcal{F}$, is the maximum number of hyperedges in an n-vertex, r-uniform, simple-hypergraph which does not contain an isomorphic copy of $\mathcal{H}$, for all $\mathcal{H} \in \mathcal{F}$, as a sub-hypergraph.*

An r-muti-hypergraph $\mathcal{H}$ is a pair $(V(\mathcal{H}), E(\mathcal{H}))$,where $V(\mathcal{H})$ is a finite set and $E(\mathcal{H})$ is a multi-set of subsets of size r of V. The same definitions for Turán number may be extended for multi-hypergraphs, we denote the Turán number for multi-hypergraphs by $\mathrm{ex}_r^{multi}(n, \mathcal{F})$.

The classical theorem of Erdős and Gallai (Theorem 1.24) was extended to Berge paths in r-uniform hypergraphs by Győri, Katona and Lemons [47].

Theorem 4.1 (Győri, Katona, Lemons [47]). *Let n, k, r be positive integers and let $\mathcal{H}$ be an r-uniform hypergraph with no Berge path of length k. If $k > r+1 > 3$, we have*

$$e(\mathcal{H}) \leq \frac{n}{k}\binom{k}{r}.$$

If $r \geq k > 2$, we have

$$e(\mathcal{H}) \leq \frac{n(k-1)}{r+1}.$$

The remaining case when $k = r+1$ was settled later by Davoodi, Győri, Methuku and Tompkins [14]. In this case the Turán number matches the upper bound of Theorem 4.1 in the $k > r+1$ case.

We now turn our attention to the case of trees in hypergraphs. The Turán number of certain kinds of trees in r-uniform hypergraphs has long been a major topic of research. For example, there is a notoriously difficult conjecture of Kalai [30] which is more general than the Erdős-Sós conjecture. The trees which Kalai considers are generalizations of the notion of tight paths in hypergraphs. In another direction, Füredi [32] investigated linear trees, constructed by adding $r-2$ new vertices to every edge in a (graph) tree. In this setting, he proved asymptotic results for all uniformities at least 4. Whereas, the articles above considered classes of trees containing tight and linear paths, respectively, we will consider the setting of Berge trees.

In the range when $k > r$, a number of results on forbidding Berge trees were obtained by Gerbner, Methuku and Palmer in [37]. In particular they proved that if we assume the Erdős-Sós conjecture holds for a tree T with k edges and all of its sub-trees and also that $k > r+1$, we have $\mathrm{ex}_r(n, \mathcal{B}T) \leq \frac{n}{k}\binom{k}{r}$ (a construction matching this bound when k divides n is given by n/k disjoint copies of the complete r-uniform hypergraph on k vertices). In the present chapter, we will consider the range $r > k$, where we prove some exact results.

4.2 Main Results

Considering multi-hypergraphs, we prove the following.

Theorem 4.2 (Győri, Salia, Tompkins, Zamora. [52]). *Let n, k, r be positive integers and let T be a k-edge tree, then for all $r \geq (k-1)(k-2)$,*

$$\mathrm{ex}_r^{multi}(n, \mathcal{B}T) \leq \frac{n(k-1)}{r}.$$

If $r > (k-1)(k-2)$ and T is not a star, equality holds if and only if r divides n and the extremal multi-hypergraph is $\frac{n}{r}$ disjoint hyperedges, each with multiplicity $k-1$. If T is a star equality holds only for all $(k-1)$-regular multi-hypergraphs.

We conjecture that Theorem 4.2 holds for the following wider set of parameters.

Conjecture 4.3. *Let n, k, r be positive integers and let T be a k-edge tree, then for all $r \geq k+1$,*

$$\mathrm{ex}_r^{multi}(n, \mathcal{B}T) \leq \frac{n(k-1)}{r}.$$

For all trees T, where T is not a star, equality holds if and only if r divides n and the extremal multi-hypergraph is $\frac{n}{r}$ disjoint hyperedges each with multiplicity $k-1$.

The special case of Conjecture 4.3, when the forbidden tree is a path, was settled by Győri, Lemons, Salia and Zamora (see Corollary 1 in [48]).

We now define a class of hypergraphs which we will need when we classify the extremal examples in our main result about simple hypergraphs, Theorem 4.6.

Definition 4.4. *An r-uniform hypergraph $\mathcal{H}$ is* two-sided *if $V(\mathcal{H})$ can be partitioned into a set X and pairwise disjoint sets A_i, $i = 1, 2, \ldots, t$ (also disjoint from X) of size $r-1$, such that every hyperedge is of the form $\{x\} \cup A_i$ for some $x \in X$. We say that a two-sided r-uniform hypergraph is (a, b)-regular if every vertex of X has degree a and every vertex of $\bigcup_{i=1}^{t} A_i$ has degree b.*

Remark 4.5. *A two-sided r-uniform hypergraph can also be viewed as a graph obtained by taking a bipartite graph G with bipartite classes X and Y, and "blowing up" each vertex of Y to a set of size $r-1$, and replacing each edge $\{x, y\}$ by the r-hyperedge containing x together with the blown up set for y.*

Theorem 4.6 (Győri, Salia, Tompkins, Zamora. [52])**.** *Let n, k, r be positive integers and let T be a k-edge tree which is not a star, then for all $r \geq k(k-2)$,*

$$\mathrm{ex}_r(n, \mathcal{B}T) \leq \frac{n(k-1)}{r+1}.$$

Equality holds if and only if $r+1$ divides n, and the extremal hypergraph is obtained from $\frac{n}{r+1}$ disjoint sets of size $r+1$, each containing $k-1$ hyperedges. Unless k is odd, and T is the balanced double star, where the balanced double star is the tree obtain from and edge by adding $\frac{k-1}{2}$ incident edges to each of the ends of the edge, in which case equality holds if and only if $r+1$ divides n and $\mathcal{H}$ is obtained from the disjoint union of sets of size $r+1$ containing $k-1$ hyperedges each and possibly a $(k-1, \frac{k-1}{2})$-regular two-sided r-uniform hypergraph (see Figure 4.1).

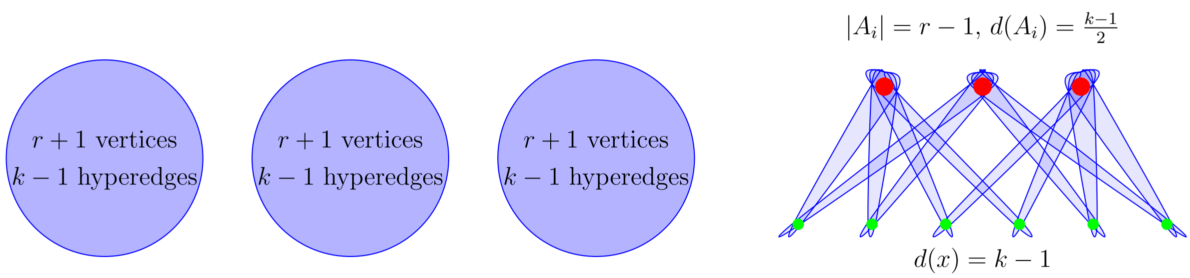


Figure 4.1: An extremal graph for Theorem 4.6 is pictured. Any such graph can be obtained from disjoint copies of a sets of $r+1$ vertices with $k-1$ hyperedges and if T is the balanced double star, possibly a $(k-1, \frac{k-1}{2})$-regular two-sided r-uniform hypergraph.

4.3 Proofs of the main results

We are going to use the following fact about trees, before proving the next bound on the degrees of the vertices in clusters.

Claim 4.7. *If T is k-edge tree which is not a star, then there exists a vertex of T which is not a leaf and has degree at most $\frac{k+1}{2}$.*

Proof. Let T' be the tree obtained by T by removing every leaf of T, since T is not a star, T' has at least two vertices, take any v, w which are leaves in T', and note that for each, every neighbor but one is a leaf, and also, since at most one of the k edges of T is incident with both u and v, we have that $d_T(u) + d_T(v) \leq k + 1$. And so, one of these vertices have the desired properties. □

Now we introduce two more definitions which we will need in the proofs.

Definition 4.8. *Let $\mathcal{H}$ be a (multi-)hypergraph. A $(k-1)$-*cluster *is a set of $k-1$ hyperedges of $\mathcal{H}$ that intersect in at least $k-1$ vertices. The intersection of the $k-1$ hyperedges is called the* core *of the $(k-1)$-cluster. The union of the $k-1$ hyperedges is called the* span *of the $(k-1)$-cluster.*

Definition 4.9. *Let $\mathcal{H} = (V, E)$ be a multi-hypergraph. A multi-hypergraph $\mathcal{H}' = (V', E')$ is called a* reduced sub-hypergraph *of $\mathcal{H}$ if $V' \subseteq V$ and there exists an injection $f : E' \to E$ such that $h \subseteq f(h)$ for all $h \in E'$. For an edge $h \in E'$ we call $f(h) \in E$ its* correspondent *edge in $\mathcal{H}$.*

In the following claims, we bound the degrees of the vertices in a $(k-1)$-cluster for a hypergraph which does not contain a copy of a Berge tree.

Claim 4.10. *Let n, k, r be positive integers, with $r \geq k+1$, and let T be a k-edge tree. If $\mathcal{H}$ is an r-uniform multi-hypergraph containing no Berge copy of T and S is a $(k-1)$-cluster in $\mathcal{H}$, then the vertices in the core of S have degree exactly $k-1$. In particular, the core vertices of S are only incident with the hyperedges of S.*

Proof. Let C be the set of vertices in the core of S. Suppose, by contradiction, there is a vertex v in C with degree at least k, and let T' be a tree obtained from T by removing any two leaves x, y. Suppose that the neighbors of these leaves are x' and y' respectively (it is possible that $x' = y'$). Since C has at least $k-1$ vertices and there are $k-1$ hyperedges containing all the vertices in C, we can greedily embed T' in C in such a way that v takes the role of x'. Suppose the vertex u takes the role of y' in this greedy embedding. We can complete the embedding of T by using the last hyperedge of S and an unused vertex in it (one exists since $r \geq k+1$) to embed y. Then since the degree of v is at least k, we have a hyperedge available to embed x as a unused vertex of this hyperedge. Thus, we have found a Berge copy of T in $\mathcal{H}$, a contradiction. □

Claim 4.11. *Let n, k, r be positive integers, with $r \geq k+1$, and let T be a k-edge tree which is not a star. If $\mathcal{H}$ is an r-uniform multi-hypergraph containing no Berge copy of T and S is a $(k-1)$-cluster of $\mathcal{H}$, then any vertex in the span of S that is incident with a hyperedge not from S, has degree at most $\left\lfloor \frac{k-1}{2} \right\rfloor$.*

Proof. Since T is not a star, by Claim 4.7, there is a vertex $x \in V(T)$ which is not a leaf and has degree s, $s \le \left\lfloor \frac{k+1}{2} \right\rfloor$, such that all but one of its neighbors is a leaf, let y be the neighbor of x which is not a leaf. Suppose, by contradiction, there is a vertex v in the span of S which is incident with a hyperedge that is not in S and v has degree at least $\left\lfloor \frac{k+1}{2} \right\rfloor$. Let C be the set of vertices in the core of S. From Claim 4.10 we know that v cannot be in C. Pick s hyperedges $h_1, h_2, \dots, h_s$ incident to v in such a way that h_1 is not in S and h_2 is in S. Choose a vertex $w \in h_1$ not in C (in fact, every vertex in h_1 is outside C by Claim 4.10) and $u \in h_2$ in C. Choose further distinct vertices $v_3, v_4, \dots, v_s$ from the hyperedges $h_3, h_4, \dots, h_s$. The vertex v will be assigned to the vertex x in the tree, and the vertex u will be assigned to the vertex y ($v_3, v_4, \dots, v_s$ will be assigned to the leaves adjacent to x). Thus, using the hyperedges $h_1, h_2, \dots, h_s$ we can embed the vertex x and all its neighbors in T using at most $s-1$ hyperedges from S and at most $s-1$ vertices from C (v and w are not in C).

There are at least $(k-1)-(s-1) = k-s$ remaining vertices in C. Each of these is contained in at least $k-s$ unused hyperedges of S. Thus, the remaining $k-s$ vertices of the tree can be mapped to distinct vertices from C, and the remaining edges of the tree may be assigned to distinct unused hyperedges of S. □

Remark 4.12. *Note that by Claim 4.10 and Claim 4.11, if $\mathcal{H}$ is a multi-hypergraph with uniformity $r \ge k+1$ that does not contain a Berge copy of a tree on k edges which is not a star, then $(k-1)$-clusters of $\mathcal{H}$ are edge-disjoint.*

Lemma 4.13. *Let k be a positive integer and let T be a k-edge tree which is not a star. Let $\mathcal{H}$ be a multi-hypergraph not necessarily uniform, not containing a Berge copy of T, and assume that each hyperedge in $\mathcal{H}$ has size at least $k+1$. If there exists a reduced sub-hypergraph $\mathcal{H}' = (V', E')$ of $\mathcal{H}$ such that $d_{\mathcal{H}'}(v) \ge k-1$ for each $v \in V'$ and $|h| \ge k-1$ for each $h \in E'$, then $\mathcal{H}'$ contains a $(k-1)$-cluster. Note that if S is a $(k-1)$-cluster in $\mathcal{H}'$, then the correspondent edges of S in $\mathcal{H}$ are a $(k-1)$-cluster.*

Proof. Let $h_2 \in E'$. We will show that every vertex in h_2 is contained in the same set of hyperedges in E'. Let $v_1, v_2 \in h_2$, and suppose by contradiction that there exists a hyperedge h_3 incident to v_2 and not to v_1. Enumerate the vertices of T by $x_0, x_1, \dots, x_k$ in such a way that the graph induced by the vertices $x_0, x_1, \dots, x_i$ is connected for all i, x_0 is a leaf of T and x_0, x_1, x_2, x_3 is a path of length 3 (such a path exists since T is not a star). For each $i = 1, 2, \dots, k$, the vertex x_i is adjacent to exactly one vertex of smaller index, call the edge using x_i and the vertex of smaller index e_i.

We can embed T into $\mathcal{H}$ in the following way. First assign v_1 to x_1, h_2 to $\{x_1, x_2\}$, v_2 to x_2, h_3 to $\{x_2, x_3\}$ and any vertex in $v_3 \in h_3 \setminus \{v_1, v_2\}$ to x_3. For $i = 4, \dots, k$, suppose $e_i = \{x_i, x_{j_i}\}$. Pick any hyperedge $h_i \in E'$ incident to v_{j_i} and distinct from $h_2, h_3, \dots, h_{i-1}$ (such hyperedges exist since $d_{\mathcal{H}'}(v_{j_i}) \ge k-1$) and assign it to e_i. If $i \le k-1$, pick any $v_i \in h_i \setminus \{v_1, v_2, \dots, v_{i-1}\}$, and if $i = k$, then let $\tilde{h}_k$ be the correspondent hyperedge of h_k in $\mathcal{H}$. As $\tilde{h}_k$ has size bigger than k, let v_k be any vertex in $\tilde{h}_k \setminus \{v_1, v_2, \dots, v_{k-1}\}$. This vertex v_k is assigned to x_k. Finally, since v_1 is incident with at least $k-1$ hyperedges distinct to h_3, there is a hyperedge h_1 incident to v_1 and distinct from the already chosen hyperedges. Let $\tilde{h}_1$ be the correspondent hyperedge of h_1. Take any vertex in $\tilde{h}_1$ which has not been assigned yet and assign it to x_0. Thus, by replacing the edge h_i with their correspondent hyperedges, we have found a Berge copy of T in $\mathcal{H}$, a contradiction.

It follows that for any $v_1, v_2 \in h_2$, we have that v_1 and v_2 must be incident with the same set of hyperedges in $\mathcal{H}'$ (by assumption at least $k-1$), and so $\mathcal{H}'$ contains a $(k-1)$-cluster. $\square$

Lemma 4.13 says that if $\mathcal{H}$ does not contain a Berge copy of a tree and we are able to find a large enough reduced sub-hypergraph, then $\mathcal{H}$ must have at least one $(k-1)$-cluster. The main idea of the proofs of the main results is to show that if $\mathcal{H}$ has too many hyperedges and no Berge copy of a tree, then after removing all $(k-1)$-clusters, we would still be able to find a large enough reduced sub-hypergraph. This would imply that there is still another $(k-1)$-cluster in $\mathcal{H}$, a contradiction.

Proof of Theorem 4.2. Let T be a k-edge tree, which is not a star. Suppose that $\mathcal{H}$ is an n-vertex r-uniform hypergraph with at least $\frac{n(k-1)}{r}$ hyperedges such that $\mathcal{H}$ does not contain a Berge copy T, and let G be the incidence bipartite graph of $\mathcal{H}$, i.e., the bipartite graph with color classes $V(\mathcal{H})$ and $E(\mathcal{H})$ where $v \in V(\mathcal{H})$ is adjacent to $h \in E(\mathcal{H})$ if and only if $v \in h$.

Since $e(\mathcal{H}) \geq \dfrac{n(k-1)}{r}$, we have

$$\frac{e(G)}{v(G)} = \frac{re(\mathcal{H})}{n+e(\mathcal{H})} = \frac{r}{\frac{n}{e(\mathcal{H})}+1} \geq \frac{r}{\frac{r}{k-1}+1} = \frac{r(k-1)}{r+k-1},$$

and note that

$$\begin{aligned} &\frac{r(k-1)}{r+k-1} \geq k-2 \\ \Leftrightarrow & r(k-1) \geq (k-2)(r+k-1) = r(k-1) + (k-2)(k-1) - r \\ \Leftrightarrow & r \geq (k-2)(k-1). \end{aligned}$$

Hence $d(G) = \frac{2e(G)}{v(G)} \geq 2\left(\frac{r(k-1)}{r+k-1}\right) \geq k-2$, since $r \geq (k-2)(k-1)$. Suppose $\mathcal{H}$ has t distinct $(k-1)$-clusters $S_1, S_2, \ldots, S_t$ (recall that by Remark 4.12 $(k-1)$-clusters are edge-disjoint). For each S_i, let X_i be the set of vertices which are incident only with hyperedges of S_i, let $X = \bigcup_{i=1}^{t} X_i$ and let Y be the set of vertices that are not in X but are incident with at least one of the $(k-1)$-clusters. Let G_1 be the induced subgraph of G obtained by removing X, Y and all $(k-1)$-cluster hyperedges from the vertex set of G. We will show that $d(G_1) \geq d(G)$ (provided G_1 is not the empty graph).

The number of edges removed in G is $\sum_{v \in X} d_{\mathcal{H}}(v) + \sum_{v \in Y} d_{\mathcal{H}}(v)$. Since the degree of each $v \in X$ is at most $k-1$, we have that $\left(\sum_{v \in X} d_{\mathcal{H}}(v)\right) \leq |X|(k-1)$. Also X is only incident with the $(k-1)$-cluster hyperedges, thus we also have the bound $\left(\sum_{v \in X} d_{\mathcal{H}}(v)\right) \leq tr(k-1)$, and since the degree of each $v \in Y$ is at most $\frac{k-1}{2}$ (Claim 4.11), we have that $\left(\sum_{v \in Y} d_{\mathcal{H}}(v)\right) \leq \dfrac{(k-1)|Y|}{2}$. Therefore

$$\left(\sum_{v \in X} d_{\mathcal{H}}(v) + \sum_{v \in Y} d_{\mathcal{H}}(v)\right)(r+k-1)$$

$$= \left(\sum_{v\in X} d_{\mathcal{H}}(v)\right) r + \left(\sum_{v\in X} d_{\mathcal{H}}(v)\right)(k-1) + \left(\sum_{v\in Y} d_{\mathcal{H}}(v)\right)(r+k-1)$$

$$\leq |X|\, r(k-1) + tr(k-1)^2 + \frac{(k-1)\,|Y|}{2}(r+k-1) \leq r(k-1)(|X| + t(k-1) + |Y|),$$

where in the last inequality we used $\frac{r+k-1}{2} < r$. Thus, equality can hold only if $Y = \emptyset$.

Rearranging we have

$$\left(\sum_{v\in X} d_{\mathcal{H}}(v) + \sum_{v\in Y} d_{\mathcal{H}}(v)\right) \leq \frac{r(k-1)}{r+k-1}\left(|X| + t(k-1) + |Y|\right). \tag{4.1}$$

The left-hand side of (4.1) is the number of removed edges, and the right-hand side is $d(G)/2$ times the number of removed vertices. Therefore, by Lemma 1.4, if G_1 is non-empty, we have that

$$d(G_1) \geq d(G) \geq 2(k-2).$$

Hence, by Lemma 1.3 there is a subgraph G_2 of G_1 with minimum degree at least $k-1$. Suppose that G_2 has bipartite classes $A \subseteq V(\mathcal{H})$ and $B \subseteq E(\mathcal{H})$, and define $\mathcal{H}'$ by taking the vertex set $V' = A$ and $E' = \{h \cap V' : h \in B\}$. The condition on the minimum degree of G_2 implies that every vertex of $\mathcal{H}'$ has degree at least $k-1$ and every hyperedge of $\mathcal{H}'$ has size at least $k-1$. Then by Lemma 4.13, $\mathcal{H}'$ contains a $(k-1)$-cluster, but this $(k-1)$-cluster corresponds to a $(k-1)$-cluster in $\mathcal{H}$ contradicting the fact that we removed every $(k-1)$-cluster from $\mathcal{H}$. So $\mathcal{H}$ must contain a Berge copy of T, unless G_1 is empty.

Note that, for G_1 to be empty it is necessary that $d(G) = 2\frac{r(k-1)}{r+k-1}$ and that equality holds in the inequality (4.1). This can be possible only if $Y = \emptyset$ and

$$|X| = \frac{1}{k-1}\sum_{v\in X} d_{\mathcal{H}}(v) = tr.$$

Since every $(k-1)$-cluster contains at least r vertices, we have $|X_i| \geq r$, and so each X_i must have size exactly r, hence $\mathcal{H}$ is the disjoint union of t hyperedges each with multiplicity $k-1$. Therefore, the number of vertices would be a multiple of r and $e(\mathcal{H}) = \frac{n(k-1)}{r}$. Hence if $e(\mathcal{H}) \geq \frac{n(k-1)}{r}$, then $\mathcal{H}$ must contain a Berge copy of T, or $r|n$ and $\mathcal{H}$ is the disjoint union of $\frac{n}{r}$ hyperedges each with multiplicity $k-1$. $\square$

Remark 4.14. *For $r = (k-2)(k-1)$, the proof above also shows that if $e(\mathcal{H}) > \frac{n(k-1)}{r}$, then $\mathcal{H}$ must contain a Berge copy of T. However, the extremal construction does not follow from that proof.*

Proof of Theorem 4.6. Let T be a k-edge tree which is not a star. We may assume $k > 3$, since otherwise T is a path, and we already know the result for paths. Let $\mathcal{H}$ be an n-vertex hypergraph with at least $\frac{n(k-1)}{r+1}$ hyperedges and $r \geq k(k-2)$. We will proceed by induction on the number of vertices n; the base cases $n \leq r+1$ are trivial.

If there is a set U of size $r+1$ which is incident with at most $k-1$ hyperedges, put $V' = V \setminus U$ and let $n' = |V'| = n-r-1$. By induction, $\mathcal{H}'$ the hypergraph induced by V', has at most $\frac{n'(k-1)}{r+1}$ hyperedges and equality holds if $r+1|n'$ and $\mathcal{H}'$ is the disjoint union of cliques, unless T is the balanced double star, then it may contain a $(k-1, \frac{k-1}{2})$-regular two-sided hypergraph as described in the statement of the theorem. Note that if one of the hyperedges incident with U is incident with a vertex v, $v \in V'$, then v has degree at least $\lfloor \frac{k+1}{2} \rfloor$, and v is in a $(k-1)$-cluster of $\mathcal{H}'$, thus we have a Berge copy of T from Claim 4.11. Hence, the $k-1$ hyperedges incident with U are contained in the vertex set U and $\mathcal{H}$ has the desired structure.

Similarly to the proof of Theorem 4.2, we have that

$$\frac{e(G)}{v(G)} = \frac{re(\mathcal{H})}{n + r(\mathcal{H})} = \frac{r}{\frac{n}{e(\mathcal{H})} + 1} \geq \frac{r}{\frac{r+1}{k-1} + 1} = \frac{r(k-1)}{r+k},$$

and note that

$$\frac{r(k-1)}{r+k} \geq k-2 \Leftrightarrow r(k-1) \geq (k-2)(r+k) = r(k-1) + (k-2)k - r \Leftrightarrow r \geq k(k-2).$$

Hence $d(G) = \frac{2e(G)}{v(G)} \geq 2\left(\frac{r(k-1)}{r+k-1}\right) \geq k-2$, since $r \geq (k-2)(k-1)$. Suppose that $\mathcal{H}$ has t distinct $(k-1)$-clusters $S_1, S_2, \ldots, S_t$. Define the sets $X_1, \ldots, X_t, X$ and Y as in the proof of Theorem 4.2. We are going to remove all vertices and hyperedges of these $(k-1)$-clusters as in the previous proof, and we will denote the incidence bipartite graph of $\mathcal{H}$ by G. By G_1 we will denote the incidence bipartite graph of the hypergraph $\mathcal{H}'$, obtained from $\mathcal{H}$ after removing the $(k-1)$-clusters.

If $|X_i| \geq r+1$ for some i, then by taking $U \subseteq X_i$ of size $r+1$, we would have that U is incident with at most $k-1$ hyperedges, and we would be done by induction. Hence we assume that $|X_i| \leq r$.

For each i, with $|X_i| = r$, we have

$$\sum_{v \in X_i} d_{\mathcal{H}}(v) \leq (r-1)(k-1) + 1 = |X_i|(k-1) - (k-2),$$

since any hyperedge is incident with at most $r-1$ vertices from X_i, with the possible exception of at most one hyperedge (X_i, if $X_i \in E(\mathcal{H})$).

For each i, with $|X_i| \leq r-1$, we have

$$\sum_{v \in X_i} d_{\mathcal{H}}(v) \leq |X_i|(k-1) \leq (r-1)(k-1).$$

Let a be the number of X_i, $1 \leq i \leq t$, with the size r. Then we have the following inequalities

$$\sum_{v \in X} d_{\mathcal{H}}(v) = \sum_{\substack{|X_i|=r \\ v \in X_i}} d_{\mathcal{H}}(v) + \sum_{\substack{|X_i|<r \\ v \in X_i}} d_{\mathcal{H}}(v) \leq t(r-1)(k-1) + a, \tag{4.2}$$

and

$$\sum_{v \in X} d_{\mathcal{H}}(v) \leq \sum_{\substack{|X_i|=r \\ v \in X_i}} (|X_i|)(k-1) - (k-2)) + \sum_{\substack{|X_i|<r \\ v \in X_i}} |X_i|(k-1) = |X|(k-1) - a(k-2). \tag{4.3}$$

We also have

$$tr(k-1) \le \sum_{v\in X} d_{\mathcal{H}}(v) + \sum_{v\in Y} d_{\mathcal{H}}(v) \le t(r-1)(k-1) + a + \frac{k-1}{2}\left|Y\right|, \qquad (4.4)$$

where in the first inequality follows from the fact the set the edges of $t(k-1)$ hyperedges of the t cluster are incident only with the set $X \cup Y$, and the second inequality follows directly from Claim 4.11 together with the fact that $d_{\mathcal{H}}(v) \le k-1$ by definition.

Rearranging (4.4) yields

$$t(k-1) \le a + \frac{\left|Y\right|(k-1)}{2}. \qquad (4.5)$$

The following three bounds come from multiplying inequality (4.3) and (4.2) by r and k, respectively, and the bound from Claim 4.11 by $k+r$.

$$\left(\sum_{v\in X} d_{\mathcal{H}}(v)\right) r \le \left|X\right| r(k-1) - ar(k-2). \qquad (4.6)$$

$$\left(\sum_{v\in X} d_{\mathcal{H}}(v)\right) k \le t(r-1)k(k-1) + ak. \qquad (4.7)$$

$$\left(\sum_{v\in Y} d_{\mathcal{H}}(v)\right)(k+r) \le \frac{\left|Y\right|(k-1)}{2}(k+r). \qquad (4.8)$$

Now we bound the number of deleted hyperedges times $r+k$. From (4.6), (4.7), (4.8) and then (4.5), it follows that

$$\begin{aligned}
&\left(\sum_{v\in X} d_{\mathcal{H}}(v) + \sum_{v\in Y} d_{\mathcal{H}}(v)\right)(k+r) \\
\le\ & \left|X\right| r(k-1) - ar(k-2) + t(r-1)k(k-1) + ak + \frac{\left|Y\right|(k-1)}{2}(k+r) \\
=\ & \left|X\right| r(k-1) - ar(k-2) + tr(k-1)^2 + t(k-1)(r-k) + ak + \frac{\left|Y\right|(k-1)}{2}(k+r) \\
\le\ & \left|X\right| r(k-1) - ar(k-2) + tr(k-1)^2 + a(r-k) + ak + \frac{\left|Y\right|(k-1)}{2}(k+r+(r-k)) \\
=\ & \left|X\right| r(k-1) - ar(k-3) + tr(k-1)^2 + \left|Y\right|(k-1)r \\
=\ & r(k-1)(\left|X\right| + \left|Y\right| + t(k-1)) - ar(k-3) \\
\le\ & r(k-1)(\left|X\right| + \left|Y\right| + t(k-1)).
\end{aligned}$$

Rearranging we have

$$\left(\sum_{v\in X} d_{\mathcal{H}}(v) + \sum_{v\in Y} d_{\mathcal{H}}(v)\right) \le \frac{r(k-1)}{r+k}\left(\left|X\right| + t(k-1) + \left|Y\right|\right). \qquad (4.9)$$

The left-hand side of (4.9) is the number of removed edges, and the right-hand side of (4.9) is $d(G)/2$ times the number of removed vertices.

Hence, by Lemma 1.4 if G_1 is nonempty, we have that

$$d(G_1) \geq d(G) \geq 2(k-2).$$

Thus, by Lemma 1.3 we can find a subgraph G_2 of G_1 with minimum degree at least $k-1$. Suppose that G_2 has bipartite classes $A \subseteq V$ and $B \subseteq E(\mathcal{H})$, define $\mathcal{H}'$ by taking the vertex set $V' = A$ and hyperedge set $E' = \{h \cap V' : h \in B\}$. The condition on the minimum degree of G_2 implies that every vertex of $\mathcal{H}$ has minimum degree at least $k-1$, and every hyperedge of $\mathcal{H}'$ has size at least $k-1$. Then by Lemma 4.13, $\mathcal{H}'$ contains a $(k-1)$-cluster, which contradicts that we have removed all $(k-1)$-clusters in $\mathcal{H}$.

For G_1 to be empty it is necessary that $d(G) = 2\frac{r(k-1)}{r+k}$, and for (4.9) to hold with equality, we must have that $e(\mathcal{H}) = \frac{n(k-1)}{r+1}$. To obtain equality in (4.9), it is necessary that $a = 0$ (since $k > 3$) and that every hyperedge contains one of the X_i. It then follows that $|X| = t(r-1)$, and by (4.5), $|Y| = 2t$. By (4.8), for every $v \in Y$, we have $d_{\mathcal{H}}(v) = \frac{k-1}{2}$, so $n = t(r+1)$. Then $\mathcal{H}$ is a disjoint union of sets of $r+1$ vertices with $k-1$ hyperedges, and a hypergraph constructed from the classes $A = \{X_1, X_2 \ldots, X_t\}$ and $B = Y$, where $\{y, X_i\}$ is an edge if $X_i \cup \{y\}$ is a hyperedge of $\mathcal{H}$. Note that $2t = 2|A| = |B|$, the degree of every vertex in B is $\frac{k-1}{2}$ and every vertex of A has degree $k-1$; that is, $\mathcal{H}$ is a $(k-1, \frac{k-1}{2})$-regular two-sided hypergraph.

However, this is only possible if k is odd, and it is simple to check that this construction contains a Berge copy of every k-edge tree which is not a balanced k-edge double star or the k-edge star. $\square$

Chapter 5

Ramsey numbers of Berge-hypergraphs and related structures

5.1 Introduction

We will recall the definition of the set of Berge-copies of a graph G. In fact, we will give a more general definition in which rather than starting with a graph G we may start with any uniform hypergraph.

Definition 5.1. *Let $\mathcal{H} = (V, \mathcal{E})$ be a k-vertex s-uniform hypergraph. Then given an integer $r \geq s$, $B\mathcal{H}$ (the set of Berge-copies of $\mathcal{H}$) is defined to be the set of r-uniform hypergraphs $\mathcal{H}' = (W, \mathcal{F})$ such that there exist $U \subseteq W$ and bijections $\phi : V \to U$, $\psi : \mathcal{E} \to \mathcal{F}$ such that for all $e = \{u_1, u_2, \ldots, u_s\} \in \mathcal{E}$, $\{\phi(u_1), \phi(u_2), \ldots, \phi(u_s)\} \subseteq \psi(e)$. In this case, we call U the core of $\mathcal{H}'$.*

Remark 5.2. *For simplicity, we will often (when it cannot lead to confusion) say that a hypergraph is a $B\mathcal{H}$ to mean it is an element of $B\mathcal{H}$. For example, we may, in a colored hypergraph, say that a certain hypergraph is a red BK_t, meaning that it is an element of the set BK_t which is red. Similar terminology will be used with respect to the other structures which we define later.*

One of the main topics of this chapter is determining the Ramsey number of the set of Berge-copies of a hypergraph (mainly in the graph case). We show that the 2-color Ramsey number of BK_t versus BK_s is linear. In particular, we prove the following theorem:

Theorem 5.3 (Salia, Tompkins, Wang, Zamora. [79])**.**

$$R^3(BK_s, BK_t) = \begin{cases} t+s-1 & \text{if } \{s,t\} = \{2\}, \{3\}, \{2,3\} \text{ or } \{2,4\}, \\ t+s-2 & \text{if } s=2,\ t \geq 5, \text{ or } s=3,\ t \geq 4 \text{ or } s=t=4, \\ t+s-3 & \text{if } s \geq 4 \text{ and } t \geq 5. \end{cases}$$

For higher uniformity, we will show the following theorem.

Theorem 5.4 (Salia, Tompkins, Wang, Zamora. [79])**.**

$$R^4(BK_t, BK_t) = \begin{cases} t+2 & \text{if } 2 \le t \le 5, \\ t+1 & \text{Otherwise.} \end{cases}$$

Moreover, for general uniformity k we prove

Theorem 5.5 (Salia, Tompkins, Wang, Zamora. [79])**.** *For $k \ge 5$ and $t \ge t_0(k)$ (for $k = 5$, $t_0 = 23$ suffices),*

$$R^k(BK_t, BK_t) = t.$$

Remark 5.6. *We remark that a similar direction (but with mostly non-overlapping results) has been pursued by two other groups independently [6, 36]. In particular, [6] is primarily concerned with non-uniform hypergraphs whereas we focus solely on the uniform case.*

In addition to Berge-hypergraphs, we consider a variety of related structures. First, we discuss a more restrictive class of hypergraphs defined from a given hypergraph $\mathcal{H}$.

Definition 5.7. *Let $\mathcal{H} = (V, \mathcal{E})$ be a k-vertex s-uniform hypergraph and let $S \subset V$. The trace of $\mathcal{H}$ on S, denoted $\mathrm{Tr}(\mathcal{H}, S)$, is the hypergraph with vertex set S and hyperedge set $\{h \cap S : h \in \mathcal{E}\}$. Then, given $r \ge s$, $T\mathcal{H}$ is defined to be the set of r-uniform hypergraphs $\{\mathcal{H}' : \mathrm{Tr}(\mathcal{H}', V(\mathcal{H})) = \mathcal{H}\}$. For each such element $\mathcal{H}' \in T\mathcal{H}$, we refer to $V(\mathcal{H})$ as the core of $\mathcal{H}'$.*

This notion originates from the idea of shattering sets and the Sauer-Shelah lemma [80, 81, 84]. This lemma provides an upper bound on the size of an n-vertex (non-uniform) hypergraph avoiding $\mathrm{Tr}(\mathcal{H}, S) = 2^S$ for all k-vertex sets S. Frankl and Pach [31] investigated the same problem with the restriction that the hypergraph is r-uniform. In the case when $\mathcal{H}$ is a (graph) cycle, $T\mathcal{H}$ was studied under the name weak β-cycle [27]. In the case of complete graphs, bounds were obtained by Mubayi and Zhao in [73]. For a survey on extremal problems for traces see [33].

We now turn our attention to an even more restrictive notion called the expansion of a hypergraph.

Definition 5.8. *Let $\mathcal{H} = (V, \mathcal{E})$ be an s-uniform hypergraph. The r-expansion $H\mathcal{H}$, for $r \ge s$, is defined to be the r-uniform hypergraph formed by adding $r - s$ distinct new vertices to every hyperedge in $\mathcal{H}$. Precisely, for each hyperedge $e \in \mathcal{E}$, define the set $U_e = \{u_{e,1}, u_{e,2}, \ldots, u_{e,r-s}\}$, and let $H\mathcal{H} = (V \cup (\cup_{e \in \mathcal{E}} U_e), \mathcal{F})$ where $\mathcal{F} = \{e \cup U_e : e \in E\}$. We call V the core of $\mathcal{H}$ and $V(\mathcal{H}) \setminus V$, the set of expansion vertices.*

If $\mathcal{H}$ is a cycle we recover the well-known notion of a linear cycle. Ramsey and Turán problems for linear cycles have been investigated intensely (see, for example [56]). The Turán problem when $\mathcal{H}$ is a complete graph was investigated in [70] and [77]. See [72] for a detailed survey of Turán problems on expansions. In this article, we investigate the 2-color Ramsey number of the 3-expansion of complete graphs K_t. By definition, a 3-expansion of complete K_t has $\binom{t}{2} + t$ vertices. Thus $R^3(HK_t, HK_t) \ge \binom{t}{2} + t$. We prove in the following theorem yielding a cubic upper bound on $R^3(HK_t, HK_s)$.

Theorem 5.9 (Salia, Tompkins, Wang, Zamora. [79]). *For $t, s \geq 2$, we have*

$$R^3(HK_t, HK_s) \leq 2st(s+t).$$

Remark 5.10. *For $t \geq s$, as a lower bound we can take a blue clique on $t + \binom{t}{2} - 1$ vertices. However, there is still a gap in the order of magnitudes of quadratic versus cubic.*

In [12], Conlon, Fox and Rödl proved the same bound for diagonal Ramsey numbers. They showed that $R^3(HK_t, HK_t) \leq 4t^3$. This bound was latter improved by Fox and Li in [28] where it was shown that $R^3(HK_t, HK_t) = O(t^2 \ln t)$.

Next, we consider another way a hypergraph can be defined from another arbitrary hypergraph called a suspension [57] (or earlier enlargement [82]). Conlon, Fox and Sudakov considered the Ramsey numbers of the 3-suspension of a graph versus a 3-uniform clique in a short section of [13].

Definition 5.11. *Let $\mathcal{H} = (V, \mathcal{E})$ be an s-uniform hypergraph. The r-suspension $S\mathcal{H}$, for $r \geq s$, is defined to be the hypergraph formed by adding a single fixed set of $r - s$ distinct new vertices to every edge in $\mathcal{H}$. Precisely, let $U = \{u_1, u_2, \ldots, u_{r-s}\}$, and define $S\mathcal{H} = (V \cup U, \mathcal{F})$ where $\mathcal{F} = \{e \cup U : e \in E\}$. We call V the core of $S\mathcal{H}$ and U the set of suspension vertices.*

For suspensions of hypergraphs, we are only able to obtain Ramsey-type bounds using standard Ramsey number techniques. In particular, we show the following.

Theorem 5.12 (Salia, Tompkins, Wang, Zamora. [79]). *For $r \geq 3$, we have*

$$(1 + o(1))\frac{\sqrt{2}}{e}t\sqrt{2}^t < R^r(SK_t, SK_t) \leq R^2(K_t, K_t) + (r-2).$$

Finally, we discuss a a class of hypergraphs defined from a graph which is larger than the class defined by a Berge-hypergraph.

Definition 5.13. *The 2-shadow of a hypergraph $\mathcal{H} = (V, \mathcal{E})$, denoted $\partial_2(\mathcal{H})$, is the graph $G = (V, E)$ where $E = \{\{x, y\} : \{x, y\} \subseteq e \in \mathcal{E}\}$. Given a graph $G = (V, E)$, define ∂G to be the set of hypergraphs $\{\mathcal{H} : E(G) \subseteq \partial_2(\mathcal{H})\}$.*

In [70], Mubayi determined the Turán number of ∂K_t in all uniformities. In this Chapter, we prove the following.

Theorem 5.14 (Salia, Tompkins, Wang, Zamora. [79]). *We have*

(1) $R^3(\partial K_2, \partial K_2) = 3$.

(2) $R^3(\partial K_2, \partial K_s) = s$ *for* $s \geq 3$.

(3) $R^3(\partial K_t, \partial K_s) = t + s - 3$ *for* $s, t \geq 3$.

(4) $R^r(\partial K_t, \partial K_s) = \max\{s, t\}$ *for* $r \geq 4$ *and* $s, t \geq r$.

Remark 5.15. *Observe that for any graph G, we have $\{HG, SG\} \subset TG \subset BG \subset \partial G$.*

Organization This Chapter is organization as follows: In Section 5.2, we give the proof of Theorems 5.3, 5.4 and 5.5. In Section 5.3, we give the proof of Theorem 5.14. In Section 5.4, we show some results on the Ramsey number of trace-cliques. In Section 5.5, we give the proof of Theorems 5.9 and 5.12.

5.2 Ramsey number of Berge-hypergraphs

To avoid tedious case analysis, some of the small cases are verified by computer. The code is available at `https://github.com/wzy3210/berge_Ramsey`. We list below the results verified by the computer.

Proposition 5.16. *We have*

(1) $R^3(BK_3, BK_4) = 5$.

(2) $R^3(BK_4, BK_5) = 6$.

(3) $R^4(BK_t, BK_t) \le t + 2$ *for* $2 \le t \le 5$.

(4) $R^4(BK_6, BK_6) \le 7$.

5.2.1 Proof of Theorem 5.3

Recall that the number $R^3(BK_s, BK_t)$ is the smallest number N such that any 2-edge-colored complete 3-uniform hypergraph (with colors blue and red) on $n \ge N$ vertices either contains a blue Berge K_s or a red Berge K_t. In this subsection, we will show that

$$R^3(BK_s, BK_t) = \begin{cases} t+s-1 & \text{if } \{s,t\} = \{2\}, \{3\}, \{2,3\} \text{ or } \{2,4\}, \\ t+s-2 & \text{if } s=2, t \ge s+3, \text{ or } s=3, t \ge s+1 \text{ or } s=t=4, \\ t+s-3 & \text{if } s \ge 4 \text{ and } t \ge 5. \end{cases}$$

Let us first deal with the cases when one of s or t is small. In particular, we prove them in the following proposition.

Proposition 5.17. *We have*

(1) $R^3(BK_2, BK_2) = 3$.

(2) $R^3(BK_2, BK_3) = 4$.

(3) $R^3(BK_3, BK_3) = 5$.

(4) $R^3(BK_2, BK_4) = 5$.

(5) $R^3(BK_4, BK_4) = 6$.

(6) $R^3(BK_2, BK_t) = t$ *when* $t \ge 5$.

(7) $R^3(BK_3, BK_t) = t + 1$ *when* $t \ge 4$.

Proof. (1) is trivial since any non-trivial edge-colored 3-uniform hypergraph contains at least 3 vertices and any edge is a BK_2. For (2), $R^3(BK_2, BK_3) > 3$ since a single red edge is a complete $K_3^{(3)}$ and is not a red BK_3. For the upper bound, suppose we have an edge-colored $K_4^{(3)}$. If it has a blue edge, we get a blue BK_2. Otherwise all of the 4 edges are red, in which case we have a red BK_3. Similar reasoning gives (4) and (6). For (3), $R^3(BK_3, BK_3) > 4$ since an edge-colored $K_4^{(3)}$ with two red and two blue edges does not have a monochromatic BK_3. Similar reasoning gives the lower bound of (5). The upper bounds of (3) and (5) follow from Lemma 5.19. For (7), we first show that $R^3(BK_3, BK_t) > t$. Let $\mathcal{H}$ be an edge-color $K_t^{(3)}$ with two special vertices v_1, v_2 such that any hyperedge containing both v_1, v_2 is blue and all other hyperedges are colored red. Observe that any blue Berge clique or red Berge clique cannot contain both v_1 and v_2. Therefore, there is no blue BK_3 or red BK_t in $\mathcal{H}$. For the upper bound, it is checked by computer that $R^3(BK_3, BK_4) = 5$ and the bound $R^3(BK_3, BK_t) \leq t+1$ ($t \geq 5$) follows from Lemma 5.19, which will be proven later. $\square$

Next, we show the lower bound in the following proposition.

Proposition 5.18. *Suppose $s, t \geq 3$. We then have*

$$R^3(BK_t, BK_s) \geq t + s - 3.$$

Proof. We will construct a 2-edge-colored complete 3-uniform hypergraph $\mathcal{H}$ on $t+s-4$ vertices without a blue BK_t and red BK_s. Let $V(\mathcal{H}) = A \cup B$ where A and B are two disjoint sets with $|A| = t-2$ and $|B| = s-2$. For all $a, a' \in A$, $b \in B$, color the hyperedge $\{a, a', b\}$ blue. For all $a \in A$, $b, b' \in B$, color the hyperedge $\{a, b, b'\}$ red. Moreover, color all triples in A blue and all triples in B red. Observe that any blue Berge clique contains at most one vertex from B and any red Berge clique contains at most one vertex from A. It follows that $\mathcal{H}$ does not contain a blue BK_t or a red BK_s. Hence $R^3(BK_t, BK_s) \geq t + s - 3$. $\square$

Before we present the proof of Theorem 5.3, we will prove the following lemma.

Lemma 5.19. *Suppose $t, s \geq 3$. Then*

$$R^3(BK_t, BK_s) \leq \max\{R^3(BK_{t-1}, BK_s), R^3(BK_t, BK_{s-1})\} + 1.$$

Proof. Without loss of generality, assume $t \geq s$. Let $\mathcal{H}$ be a 2-edge-colored complete 3-uniform hypergraph with $N := \max\{R^3(BK_{t-1}, BK_s), R^3(BK_t, BK_{s-1})\} + 1$ vertices, and let V be the set of vertices. We want to show that $\mathcal{H}$ contains either a blue BK_t or a red BK_s as a sub-hypergraph.

Take $v \in V$ and let $\mathcal{H}'$ be the hypergraph induced by the vertices $V' := V \backslash \{v\}$. Since $|V'| \geq R^3(BK_{t-1}, BK_s)$, it follows by definition that $\mathcal{H}'$ contains a blue BK_{t-1} or a red BK_s. If there is a red BK_s we are done. Otherwise suppose we have a blue BK_{t-1}, with the vertex set Y as its core. Now let us consider G, the blue trace of v in $\mathcal{H}$, i.e., G is the graph with vertex set V' such that there exists an edge $\{x, y\}$ in G if and only if the hyperedge $\{x, y, v\}$ in $\mathcal{H}$ is colored blue.

Claim 5.20. *Either we can extend Y using v to obtain a blue BK_t or there exists a vertex $u \in Y$ with $d_G(u) \leq 1$. Moreover if $d_G(u) = 1$ and $\{u, w\}$ is the only edge containing u, then $d_G(w) < N - 2$.*

Proof. Consider the incidence graph of G, i.e., the bipartite graph $I = Y \cup E(G)$ such that for every $u \in Y$, $e \in E(G)$, u is incident to e if and only if $u \in e$. Observe that Y is the core of a blue BK_{t-1} with none of its hyperedges containing v. Therefore, by our definition of G (the blue trace of v in $\mathcal{H}$), if there is a matching of Y in I, then we can use the edges of the matching to obtain a blue BK_t with $Y \cup \{v\}$ as its core.

Now assume I does not contain a matching of Y. We first claim that there exists a vertex $u \in Y$ with $d_G(u) \leq 1$. Note that the degree of each $e \in E(G)$ is at most 2. Thus, if $d_I(u) \geq 2$ for all $u \in Y$, then it follows that for every $S \subseteq Y$, $|N_I(S)| \geq |S|$, which gives us a matching on Y by Hall's condition. Thus by contradiction, we have a vertex in Y of degree at most 1 in G.

Suppose now $d_G(u) = 1$ for some u in Y and $e = \{u, w\}$ is the unique edge containing u. We claim that $d_G(w) < N - 2$. Suppose by contradiction that $d_G(w) \geq N - 2$. This implies that $\{v, w, z\}$ is a blue edge for every $z \in V(\mathcal{H}) \backslash \{v, w\}$. Moreover, by our lower bound in Proposition 5.17 (when s, t are small) and Proposition 5.18, there exists another vertex $y \in V' \backslash Y$. It follows that we can extend Y into the core of a blue BK_t with the following embedding: for each $z \in Y \backslash \{w\}$, embed $\{v, z\}$ to the hyperedge $\{v, z, w\}$. Then embed $\{v, w\}$ to $\{v, w, y\}$. Thus if we do not have a blue BK_t with $Y \cup v$ as its core, then we have $d_G(w) < N - 2$. □

This claim says that either there exists $u \in Y$ such that $\{v, u, x\}$ is red for every $x \in V' \backslash \{u\}$, or there exists $u, w \in V'$ such that $\{v, u, x\}$ is red for every $x \neq w$ and there exists w_x such that $\{v, w, w_x\}$ is red. Note that the second case covers the first case by taking any $w \neq u$ in V' and picking $w_x = u$. So it suffices to assume the second case.

Now since $N - 1 \geq R^3(BK_t, BK_{s-1})$, it follows that $\mathcal{H}'$ either contains a blue BK_t or a red BK_{s-1}. We are done in the former case. So, suppose that $\mathcal{H}'$ contains a red BK_{s-1}. We will show that we can extend this BK_{s-1} by adding the vertex v into its core. Let X be the core of the Berge-K_{s-1}. Now for every $x \in X$ with $x \notin \{u, w\}$, we know that the edge $\{v, u, x\}$ is colored red. Hence we can embed $\{v, x\}$ into the red hyperedge $\{v, u, x\}$. It follows that we have an embedding of the edges from v to all but at most two vertices of X, namely u, w. In the case that $w \in X$, we can embed $\{v, w\}$ into the hyperedge $\{v, w, w_x\}$, which is red. Now if $u \notin X$, we are done. Otherwise, assume $u \in X$. Note that by the lower bounds in Proposition 5.17 (when s, t are small) and Proposition 5.18, $|V'| = N - 1 \geq \max\{R^3(BK_{t-1}, BK_s), R^3(BK_t, BK_{s-1})\} \geq s + 1$. Hence it follows that there exists another vertex $y \in V(\mathcal{H}') \backslash (X \cup \{w\})$. Note that by our choice of u, $\{v, u, y\}$ is red. Thus, we can embed $\{v, u\}$ into $\{v, u, y\}$. The above embedding extends X into the core of a red BK_s and we are done. □

Lemma 5.21. *$R^3(BK_4, BK_t) = t + 1$ for $t \geq 5$.*

Proof. We will proceed by induction on t. The base case that $R^3(BK_4, BK_5) = 6$ is verified by computer. Suppose now that Lemma 5.21 is true for all $5 \leq t' < t$. Let $\mathcal{H}$ be a 2-edge-colored complete 3-uniform hypergraph on $t + 1$ vertices. Note that by Proposition 5.17, we have $R^3(BK_3, BK_t) = t + 1$. Hence $\mathcal{H}$ either contains a blue BK_3 or a red BK_t. If the latter happens, we are done. So, suppose $\mathcal{H}$ contains a blue BK_3, with the vertex set Y as its core. Note that $t + 1 \geq 7$ and a Berge-triangle contains at most 6 vertices. Hence there exists a vertex v that is not used by any hyperedge in the blue BK_3. Similar to Lemma 5.19, let G be the blue trace of v in $\mathcal{H}$. Again, by Claim 5.20, either we can extend Y using v to obtain a blue BK_4 or there exists a vertex $u \in Y$ with $d_G(u) \leq 1$. Moreover, if $d_G(u) = 1$ and $\{u, w\}$ is the only edge containing u, then $d_G(w) < t - 1$. In the former case, we are done. Otherwise, WLOG, assume that there exists a $u \in Y$ and $w \in V(\mathcal{H}) \backslash \{v, u\}$ such that $\{v, u, x\}$ is red for every $x \neq w$ and there exists some vertex w_x such that $\{v, w, w_x\}$ is red. By induction, $\mathcal{H}[V(\mathcal{H}) \backslash \{v\}]$ contains either a blue BK_4 or a red BK_t. In the former case, we are done. In the latter case, we can extend the red BK_t to a red BK_{t+1} in the same way as in Lemma 5.19. □

Now this result together with Lemma 5.19 allows us to show the following proposition.

Proposition 5.22. *$R^3(BK_t, BK_s) \leq t + s - 3$, for $t, s \geq 4$ and $\max\{s, t\} \geq 5$.*

Proof. We already know this is true if one of t or s is 4, and so for $t, s \geq 5$ the result follows from induction on $t + s$, using Lemma 5.19. □

Theorem 5.3 follows from Proposition 5.17, 5.18 and 5.22.

5.2.2 Proof of Theorem 5.4

In this section, for ease of reference, sometimes we use the notation $h \to e$ to denote that the hyperedge $h \in E(\mathcal{H})$ is mapped to the vertex pair $e \in E(G)$ when constructing the embedding of $E(G)$ in $E(\mathcal{H})$.

Let us first deal with Theorem 5.4 for small values of t.

Proposition 5.23. *For $2 \leq t \leq 5$, $R^4(BK_t, BK_t) = t + 2$.*

Proof. For the lower bound, we use the fact that if $R^4(BK_t, BK_t) = n$, for some t, then $\binom{n}{4} \geq 2\binom{t}{2} - 1$. For $2 \leq t \leq 5$, this shows that $R^4(BK_t, BK_t) \geq t + 2$. The upper bound that $R^4(BK_t, BK_t) \leq t + 2$ for $2 \leq t \leq 5$ is verified by computer. □

Now we want to show that $R^4(BK_t, BK_t) = t + 1$ for all $t \geq 6$. Again, we start with the lower bound by showing the following proposition.

Proposition 5.24. *$R^4(BK_t, BK_t) \geq t + 1$ for all $t \geq 6$.*

Proof. We want to construct a 2-edge-coloring of a complete 4-uniform hypergraph on t vertices without a monochromatic BK_t. Let $\mathcal{H}$ be a $K_t^{(4)}$ with two special vertices v_1, v_2. Any hyperedge containing both v_1, v_2 is colored blue. All other hyperedges are colored red. We claim that there is no monochromatic BK_t in $\mathcal{H}$. Indeed, there is no red BK_t since only one of v_1, v_2 can be in any red BK_t. For blue BK_t, note that by our coloring there are only $\binom{t-2}{2}$ blue edges, which are fewer than the $\binom{t}{2}$ edges needed for BK_t. □

Now let us move on to the upper bound.

Lemma 5.25. *For $t \geq 6$, we have that*

$$R^4(BK_t, BK_t) \leq t + 1.$$

Proof. We prove the lemma by inducting on t. The base case that $R^4(BK_6, BK_6) \leq 7$ is verified by computer. Now assume that $t \geq 7$ and the lemma is true for all $t' < t$.

Let $\mathcal{H}$ be a 2-edge-colored complete 4-uniform hypergraph on a vertex set V of size $t + 1$. For ease of reference, given a set of vertices S, let $d_b(S)$ and $d_r(S)$ denote the number of blue and red hyperedges containing S as subset, respectively.

Claim 5.26. *Suppose $\mathcal{H}$ does not contain a monochromatic BK_t. Let v be a fixed vertex in $\mathcal{H}$. If there is a monochromatic BK_{t-1} (without loss of generality, assume it is blue) without using any hyperedge containing v, then there exists another vertex u such that $d_b(\{v, u\}) \leq 2$, i.e., all hyperedges containing both v, u are red except for at most two.*

Proof. Let $\mathcal{H}_b$ be the blue Berge-K_{t-1} hypergraph not using any hyperedge containing v. Let $\{u_1, u_2, \ldots, u_{t-1}\}$ be the core of $\mathcal{H}_b$. Construct a bipartite graph $G = A \cup B$ where $A = \{u_1, \ldots, u_{t-1}\}$ and $B = \binom{V \setminus \{v\}}{3}$. For $u_i \in A$, $S \in B$, u_i is adjacent to S in G if and only if $u_i \in S$ and $\{v\} \cup S$ is a blue edge in $\mathcal{H}$. Note that for every $S \in B$, $d_G(S) \leq 3$. Therefore, if $d_G(u_i) \geq 3$ for every $u_i \in A$, then there exists a matching of A in G by Hall's theorem, which implies that we can extend $\mathcal{H}_b$ to a blue BK_t by adding v into the core of $\mathcal{H}_b$. This contradicts our assumption that $\mathcal{H}$ does not have a monochromatic BK_t, and the proof of Claim 5.55 is complete. $\square$

Now for every $v \in V$, there exists a monochromatic BK_{t-1} in $\mathcal{H}[V \setminus \{v\}]$ by induction. Hence by Claim 5.26, for every vertex v, there exists another vertex u in V, such that $d_c(\{v, u\}) \geq \binom{t-1}{2} - 2$, for some $c \in \{\text{blue, red}\}$. We then call the pair $\{v, u\}$ a *c couple* where $c \in \{\text{blue, red}\}$. Moreover, call $\{a, b\}$ a 'bad pair' of $\{v, u\}$ if the hyperedge $\{a, b, v, u\}$ is not in color c.

By Claim 5.26, every vertex is contained in a couple. It follows that we have at least $(t+1)/2 \geq 4$ couples so at least two of them are of the same color. Without loss of generality, let $\{v_1, u_1\}$ and $\{v_2, u_2\}$ be two red couples. Our goal is to obtain a red embedding of a BK_t using mostly edges containing $\{v_1, u_1\}$ and $\{v_2, u_2\}$. We assume that $\{v_1, u_1\} \cap \{v_2, u_2\} = \emptyset$ and remark that the other case is similar and simpler. Let $\{a_1, b_1\}, \{a_2, b_2\}$ be the two possible bad pairs of $\{v_1, v_2\}$, and let $\{c_1, d_1\}$, $\{c_2, d_2\}$ be two possible bad pairs of $\{v_2, u_2\}$. If $\{v_1, u_1\}$ has exactly two bad pairs, we can assume that for at least one of them (with loss of generality the pair $\{a_2, b_2\}$) there is a red edge h containing it. Otherwise $\{a_1, b_1\}$ and $\{a_2, b_2\}$ are blue couples with no bad pairs and it is easy to find a blue BK_t by only using the blue edges containing $\{a_1, b_1\}$ and $\{a_2, b_2\}$.

If $\{v_1, u_1\}$ has exactly one bad pair, let $\{a_1, b_1\}$ be that pair and pick $\{a_2, b_2\}$ arbitrarily. Note that $\{a_2, b_2\}$ is contained in some red edge h. If $\{v_1, u_1\}$ has no bad pair, then pick $\{a_1, b_1\}$ and $\{a_2, b_2\}$ arbitrarily. Moreover, we assume that $\{v_1, u_1, v_2, u_2\}$ is a red edge and observe that otherwise constructing the embedding is easier.

Suppose $\{a_1, b_1\}$ and $\{a_2, b_2\}$ have a common vertex u. If $u \notin \{v_2, u_2\}$, relabel a_1, b_1 such that $a_1 = u$, and if $u \in \{v_2, u_2\}$ relabel u_2, v_2, a_1, b_1 such that $b_1 = u_2 = u$. Otherwise just relabel a_1, b_1 such that $a_1 \notin \{v_2, u_2\}$. Let $x_1, x_2, \ldots, x_{t-4}$ be an enumeration of $V' := V \setminus \{v_1, v_2, u_1, u_2, a_1\}$. If $b_1 \notin \{v_2, u_2\}$, assume $x_1 = b_1$. Otherwise assume without loss of generality that $b_1 = u_2$. We are going to construct the embedding in three phases:

Phase 1: Embed all vertex pairs in V'.

Consider the following embedding: For $i, j \in \{1, \ldots, t-4\}$, embed $\{x_i, x_j\}$ in $\{u_1, v_1, x_i, x_j\}$ if $i + j$ is odd, or in $\{u_2, v_2, x_i, x_j\}$ otherwise.

We have a red BK_{t-4} except possibly for at most three missing edges. Without loss of generality, let $\{x_{i_1}, x_{j_1}\}$, $\{x_{i_2}, x_{j_2}\}$, $\{x_{i_3}, x_{j_3}\}$ be the three possible bad pairs where $i_1 + j_1$ is odd and both $i_2 + j_2$ and $i_3 + j_3$ are even. If $\{x_{i_1}, x_{j_1}\}$ is indeed a bad pair of $\{v_1, u_1\}$, then it follows that $\{x_{i_1}, x_{j_1}\} = \{a_2, b_2\}$. Then we can embed $\{x_{i_2}, x_{j_2}\}$ in $\{v_1, u_1, x_{i_2}, x_{j_2}\}$, embed $\{x_{i_3}, x_{j_3}\}$ in $\{v_1, u_1, x_{i_3}, x_{j_3}\}$ and embed $\{x_{i_1}, x_{j_1}\}$ in h. Otherwise, $\{x_{i_1}, x_{j_1}\}$ does not exist and the above embedding still works except when one of $\{x_{i_2}, x_{j_2}\}, \{x_{i_3}, x_{j_3}\}$ is the pair $\{a_2, b_2\}$. We can then use h to embed $\{a_2, b_2\}$.

Phase 2: Embed all edges from $\{v_1, u_1, v_2, u_2\}$ to vertices in V'.

Consider the following embedding:

$$\{v_1, u_1, a_1, x_i\} \to \{x_i, u_1\} \text{ for } i \neq 1.$$
$$\{v_1, u_1, v_2, x_i\} \to \{x_i, v_1\} \text{ for } i \neq 1.$$
$$\{v_2, u_2, a_1, x_i\} \to \{x_i, u_2\}.$$
$$\{v_1, v_2, u_2, x_i\} \to \{x_i, v_2\}.$$

Note that x_1 can only be contained in one bad pair otherwise we would have picked x_1 to be a_1. Hence among the three edges $\{v_1, u_1, x_1, v_2\}$, $\{v_1, u_1, x_1, u_2\}$, $\{v_1, u_1, a_1, x_1\}$, at least two of them are red. Embed $\{x_1, v_1\}$, $\{x_1, u_1\}$ into those two red edges. If all three are red, do not use $\{v_1, u_1, u_2, x_1\}$ in this part of the embedding.

Now let us analyze the potential bad cases. There are at most 3 of these edges in Phase 2 that are not red.

If $\{u_1, v_1, a_1, x_i, \}, i \neq 1$ is blue, then use the edge $\{v_1, u_1, u_2, x_i\}$ to embed $\{u_1, x_i\}$.

If $\{v_1, u_1, v_2, x_i\}, i \neq 1$ is blue, then use the edge $\{v_1, u_1, u_2, x_i\}$ to embed $\{v_1, x_i\}$.

If there are two different indexes i, j such that $h_1 \in \{\{v_2, u_2, a_1, x_i\}, \{v_1, v_2, u_2, x_i\}\}$ and $h_2 \in \{\{v_2, u_2, a_1, x_j\}, \{v_1, v_2, u_2, x_j\}\}$ are both blue, then we can replace h_1 with $\{u_1, v_2, u_2, x_i\}$ and replace h_2 with $\{u_1, v_2, u_2, x_j\}$. The same embedding works if there is only one bad pair of $\{v_2, u_2\}$ in this phase.

If for some i both edges $\{v_1, v_2, u_2, x_i\}, \{v_2, u_2, a_1, x_i\}$ are blue, then it follows that the edge $\{v_2, u_2, x_i, y\}$ is red for every vertex y, with $y \notin \{v_1, a_1, v_2, u_2, x_i\}$. Consider the set of edges $E_i = \{\{v_2, u_2, x_i, y\} : y \notin \{v_1, v_2, u_2, a_1, x_i\}\}$. Note that $|E_i| = t - 4$. In Phase 1, at most $\lceil (t-6)/2 \rceil$ edges in E_i are used except when t is even and i is odd, in which case $\lfloor (t-6)/2 \rfloor$ edges in E_i are used. If t is even and i is odd, we have at least $t - 4 - \lfloor (t-6)/2 \rfloor \geq 3$ edges in E_i still available. In other cases, we have at least $t - 4 - \lceil (t-6)/2 \rceil \geq 2$ edges in E_i still available. Either there exist two edges in E_i that can be used to embed $\{v_2, x_i\}$ and $\{u_2, x_i\}$, or in Phase 1 there exists some j such that $\{v_1, u_1, x_i, x_j\}$ is blue and $\{v_2, u_2, x_i, x_j\}$ is used to embed $\{x_i, x_j\}$. In this case, there exists some $k \in \{1, \ldots t-4\} \backslash \{i\}$ such that $i + k$ is even and $\{v_1, u_1, x_i, x_k\}$ is red. Embed $\{x_i, x_k\}$ into $\{v_1, u_1, x_i, x_k\}$. It follows that we again have two available red edges containing x_i, v_2, u_2 to embed $\{v_2, x_i\}$, $\{u_2, x_i\}$.

Phase 3: Embed the edges in $\binom{\{u_1, v_1, u_2, v_2\}}{2}$.

If the edge $\{u_1, v_1, v_2, a_1\}$ is red, then use it to embed $\{v_1, v_2\}$. Otherwise we know that $\{v_2, a_1\}$ and $\{u_2, a_1\}$ are the two bad pairs of $\{v_1, u_1\}$. It follows that the edge $\{v_1, u_1, u_2, x_1\}$ is still available and the edge $\{v_1, u_1, v_2, x_1\}$ was used to embed x_1 with one of v_1 or u_1 (without loss of generality, assume v_1). In this case, embed $\{v_1, x_1\}$ in $\{v_1, u_1, u_2, x_1\}$ instead and use the edge $\{v_1, u_1, v_2, x_1\}$ to embed $\{v_1, v_2\}$. Now we will embed $\{v_1, u_2\}$ and $\{u_1, u_2\}$. Let $E_{u_2} = \{\{v_1, u_1, u_2, y\} : y \notin \{v_1, u_1, v_2, u_2\}\}$. Note that $|E_{u_2}| = t - 3$ and at most 2 edges in E_{u_2} are blue. Hence at least $(t-3) - 2 \geq 2$ of the edges in E_{u_2} are red. For each red edge in E_{u_2}, if it was used, it was because there exists some bad pair of $\{v_1, u_1\}$ which did not use u_2. That in turn implies that there are still at least 2 edges in E_{u_2} that are red and available. Hence we can embed $\{v_1, u_2\}$ and $\{u_1, u_2\}$ into these two edges. Similarly we can find an edge of the form $\{v_2, u_1, u_2, y\}$ to embed $\{u_1, v_2\}$.

Finally, by counting the edges used, it is easy to check that there are still red edges of the form $\{v_1, u_1, x, y\}$ and $\{v_2, u_2, x, y\}$ available to embed both $\{v_1, u_1\}$ and $\{v_2, u_2\}$, since each pair is in at least $\binom{t-1}{2} - 2$ red edges. □

To determine the Ramsey number in the case of cliques of different sizes, we first have the following bounds which are trivial from Theorem 5.4.

Proposition 5.27. *Suppose $t \geq s \geq 2$ and $t \geq 6$, then*

$$t \leq R^4(BK_t, BK_s) \leq t + 1.$$

Proof. The construction is trivial, we just take a clique on $t - 1$ vertices. The upper bound follows since $s \leq t$ implies $R^4(BK_t, BK_s) \leq R^4(BK_t, BK_t)$. □

For $s = t - 1$ we obtain the same bound as the case $s = t$.

Proposition 5.28. $R^4(BK_t, BK_{t-1}) = t + 1$ *for* $t \geq 6$.

Proof. The same construction works as the $R^4(BK_t, BK_t)$ case, and the upper bound follows from $R^4(BK_t, BK_{t-1}) \leq R^4(BK_t, BK_t)$. □

Theorem 5.29 (Salia, Tompkins, Wang, Zamora. [79])**.** *Assume $2 \leq s \leq t - 2$, and $t \geq 34$, then $R^4(BK_t, BK_s) = t$.*

Proof. In a red-blue coloring of a hypergraph $\mathcal{H}$, given a pair of vertices $\{v, u\}$, we define its blue degree to be $d_B(\{v, u\}) = |\{h \in E(\mathcal{H}) : \{v, u\} \subseteq h \text{ and } h \text{ is blue}\}|$. The red degree $d_R(\{v, u\})$ is defined analogously. Let

$$\delta_B^{(2)} = \min_{\{v,u\} \in \binom{V(\mathcal{H})}{2}} d_B(\{v, u\}),$$

and define $\delta_R^{(2)}$ similarly.

Call $\{v, u\}$ a c couple, $c \in \{blue, red\}$, if all but at most 5 of the hyperedges $\{v, u, x, y\}$ are c colored, and also call a pair $\{x, y\}$ a bad pair of the c couple $\{v, u\}$ if the hyperedge $\{v, u, x, y\}$ is not colored c.

Note that if $\delta_B^{(2)} = 0$ then we can find a pair $\{v, u\}$ such that $\{v, u, x, y\}$ is red for all x, y, and therefore there is a red BK_{t-2}. So we can assume $\delta_B^{(2)} \geq 1$.

Claim 5.30. *Suppose there are two blue couples, then either we can find a blue BK_t or we can find two red couples such that each have at most 4 bad pairs.*

Proof. Assume we have two disjoint blue couples $\{u_1, v_1\}$ and $\{u_2, v_2\}$, the case where these pairs are not disjoint is similar and simpler, and enumerate the other $t-4$ vertices as $x_1, x_2, \ldots, x_{t-4}$. Now let us do a preliminary embedding, for $i, j \in [t-4]$ use $\{u_1, v_1, x_i, x_j\}$ to embed $\{x_i, x_j\}$ when $i + j$ is odd and $\{u_2, v_2, x_i, x_j\}$ otherwise. If $i + j$ is odd and in this part of the embedding we used a red edge $\{u_1, v_1, x_i, x_j\}$ to embed $\{x_i, x_j\}$, but the edge $\{u_2, v_2, x_i, x_j\}$ is blue, then use the edge $\{u_2, v_2, x_i, x_j\}$ instead. If $i + j$ is even and in this part of the embedding we used a red edge $\{u_2, v_2, x_i, x_j\}$ to embed $\{x_i, x_j\}$, but the edge $\{u_1, v_1, x_i, x_j\}$ is blue, then use the edge $\{u_1, v_1, x_i, x_j\}$ instead. Let us call such a change to the embedding a swap. If both edges $\{u_1, v_1, x_i, x_j\}$ and $\{u_2, v_2, x_i, x_j\}$ are red or blue, then we do not change anything.

Note that at this point we have embedded a BK_{t-4} such that every edge is blue except at most five edges, in particular the possible pairs which are simultaneously bad pairs of $\{u_1, v_1\}$ and $\{u_2, v_2\}$.

Let $e_1, e_2, \ldots, e_k$ be these common bad pairs, $k \leq 5$. We begin with a simple remark which we will use again later.

Remark 5.31. *If $k \leq 1$ we could complete the embedding in such a way that each pair is contained in at least 1 blue edge.*

If $k \geq 2$ and all but at most one e_i is in at least 5 blue edges, then we can greedily embed the edges, starting from the one that is in less than 5 blue edges, since each is in at least one unused blue edge. So we can either find two of the e_i which are in at most 4 blue edges and the claim is proven or we complete the embedding of a blue BK_{t-4}, and if that is the case we will see we can complete this embedding to a blue BK_t.

Since for any fixed i, there are at most $\lceil \frac{t-4}{2} \rceil$ indices j such that $i + j$ is odd and also x_i can be in at most 10 bad pairs of $\{u_1, v_1\}$ or $\{u_2, v_2\}$, it follows that for every $i \in [t-4]$ there are at least $t-5-\lceil \frac{t-4}{2} \rceil - 10 \geq 4$ values of $j \in [t-4]$ not used in the previous steps of the embedding such that the edge $\{u_1, v_1, x_i, x_j\}$ is blue. Then again by Hall's Theorem in the incidence graph with components $X = \{\{x_i, v_2\} : i \in [t-4]\} \cup \{\{x_i, u_2\} : i \in [t-4]\}$ and Y the set of blue edges in $\{\{x_i, x_j, u_2, v_2\} : 1 \leq i < j \leq t-4\}$, we can find an embedding of the edges $\{x_i, v_2\}$ and $\{x_i, u_2\}$ for $i \in [t-4]$, and similarly we can find an embedding of the edges $\{x_i, v_1\}$ and $\{x_i, u_1\}$ for $i \in [t-4]$.

We have not yet used the hyperedges of the form $\{v_1, u_1, v_2, y\}$; there are at least $t - 8 \geq 26$ of these which are blue, and we can use them to embed $\{v_1, u_1\}, \{v_1, v_2\}$ and $\{u_1, v_2\}$. Similarly we can embed $\{v_2, u_2\}, \{u_1, u_2\}$ and $\{u_1, u_2\}$. Therefore either we can complete the matching or we find two pairs e_1, e_2 which are red couples, with at most 4 bad pairs. This completes the proof of Claim 5.30. □

Claim 5.32. *Suppose there are two red couples such that at least one has at most 4 bad pairs, then either we can find a red BK_{t-2} or we can find two blue couples such that each have at most 1 bad pair.*

Proof. Again we will assume the red couples are disjoint. Let $\{u_1, v_1\}$ and $\{u_2, v_2\}$ be couples such that $\{u_1, v_1\}$ has at most 4 bad pairs, and let $\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}, \{a_4, b_4\}$ be the bad pairs of $\{u_1, v_1\}$. Suppose these pairs are arranged by their red degree in increasing order. Let $V' = V \setminus \{v_1, v_2, u_1, u_2, a_1, a_2\} = \{x_1, x_2, \ldots, x_{t-6}\}$. Let us consider the following embedding which is similar to the one used in the previous claim: For $i, j \in [t-6]$ use $\{u_1, v_1, x_i, x_j\}$ to embed $\{x_i, x_j\}$ when $i + j$ is odd and $\{u_2, v_2, x_i, x_j\}$ otherwise. Similarly as in Claim 5.30, if we encounter a bad pair of one couple but not the other, then we can change the embedding to use more red edges, and at the end we have an embedding of a BK_{t-6} with almost every edge red, the only possible exceptions are the common bad pairs of $\{u_1, v_1\}$ and $\{u_2, v_2\}$ in V'. Hence here we have at most two ($\{a_3, b_3\}$ and $\{a_4, b_4\}$). If the red degree of these edges is at least 2, then we can greedily embed these two in these pairs to complete a red clique on V'. Otherwise one of these, and by the ordering also $\{a_1, b_1\}$ and $\{a_2, b_2\}$, will be in at most 1 red pair.

Similarly as in the proof of Claim 5.30, we use Hall's theorem to embed $\{x_i, v_2\}$, $\{x_i, u_2\}$, $\{x_i, v_1\}$ and $\{x_i, u_1\}$ for $i \in [t-6]$ (here the number $t-5-\lceil\frac{t-4}{2}\rceil-10$ is replaced by $t-7-\lceil\frac{t-6}{2}\rceil-8$, which is at least 5).

Since $\{v_1, u_1, v_2, y\}$ is red for at least $t-7 \geq 29$, and these hyperedges have not been used yet, it follows that we have enough hyperedges to embed $\{v_1, u_1\}, \{v_1, v_2\}$ and $\{u_1, v_2\}$ and similarly we can embed $\{v_2, u_2\}, \{v_1, u_2\}$ and $\{u_1, u_2\}$. $\square$

Note that if there is at most one blue couple, say $\{v, u\}$, we may put $V' = V \backslash \{u\}$ and for every pair $x, y \in V'$ the red degree of $\{x, y\}$ is at least 6. Then by Hall's Theorem, we can find a red BK_{t-1}. So we can assume there are at least two blue couples. Thus, by Claim 5.30 either we find a blue BK_t or we have two red couples such that at least one has at most 4 bad pairs, the conditions of Claim 5.32. From here we either find a red BK_{t-2} or satisfy conditions stronger than those of Claim 5.30. In this case, there is at most one shared bad pair and so we would be able to find a blue BK_t by Remark 5.31. $\square$

5.2.3 Proof of Theorem 5.5

In this short section, we will show that $R^k(BK_t, BK_t) = t$ when t is sufficiently large.

Claim 5.33. *If for all $v, u \in V$, there are at least $\binom{k}{2}$ red distinct hyperedges containing both v and u, then $\mathcal{H}$ contains a red BK_t.*

Proof. Consider the bipartite graph G with vertex set $V(G) = A \cup B$, where $A = \binom{V(\mathcal{H})}{2}$ and B is the set of all hyperedges of $\mathcal{H}$. For $a \in A$, $h \in B$, a is adjacent to h in G if and only if $a \subset h$ and h is colored red in $\mathcal{H}$. Note that for every $h \in B$, $d_G(h) \leq \binom{k}{2}$. Hence, if for all $\{v, u\} \in A$, $d_G(\{v, u\}) \geq \binom{k}{2}$, then by Hall's theorem we have a matching of A in G, which implies the existence of a red BK_t in $\mathcal{H}$. $\square$

Claim 5.34. *If $\binom{t-4}{k-4} \geq 2\binom{k}{2} - 1$, then $R^k(BK_t, BK_t) \leq t$.*

Proof. If the condition in Claim 5.33 does not hold, then there exist two vertices $v, u \in V(\mathcal{H})$ such that all but at most $\binom{k}{2} - 1$ hyperedges containing both v and u are blue. We claim that there exists a copy of a blue BK_t in $\mathcal{H}$ using only blue hyperedges containing both v and u. Consider again the bipartite graph G with vertex set $V(G) = A \cup B$, where $A = \binom{V(\mathcal{H})}{2}$ and B is the set of blue hyperedges of $\mathcal{H}$ containing both v and u. Note that for every $a \in A$ there are at least $\binom{t-4}{k-4} - \binom{k}{2} + 1 \geq \binom{k}{2}$ blue hyperedges containing a, and again by Hall's theorem we have a blue BK_t. $\square$

Using Claim 5.34, we show that $R^k(BK_t, BK_t) = t$ when $k \geq 5$ and t sufficiently large. We did not make an attempt to find the best possible constant.

Corollary 5.35. *We have*

(1) $R^5(BK_t, BK_t) = t$ *when* $t \geq 23$.

(2) $R^6(BK_t, BK_t) = t$ *when* $t \geq 13$.

(3) $R^7(BK_t, BK_t) = t$ *when* $t \geq 12$.

(4) $R^k(BK_t, BK_t) = t$ *when* $k \in \{8, 9, 10\}$ *and* $t \geq k + 4$.

(5) $R^k(BK_t, BK_t) = t$ *when* $k \geq 11$ *and* $t \geq k + 3$.

Remark 5.36. *Note that for* $k \geq 11$*, this result is sharp since for* $t = k+2$ *we have that* $\binom{t}{r} \leq 2\binom{t}{2} - 2$*. Hence* $R^k(BK_t, BK_t) \geq r + 3$.

5.2.4 Superlinear lower bounds for sufficiently many colors

In this subsection we show that for all uniformities and for sufficiently many colors, the Ramsey number for a Berge t-clique is superlinear. We start with the case $r = 3$.

Claim 5.37. *For any* $\epsilon < 1$ *we have* $R^3_3(BK_t, BK_t, BK_t) \geq (t-1)t^\epsilon$ *for* t *sufficiently large.*

Proof. Let $\epsilon < 1$. Take a vertex set consisting of the disjoint union of $t-1$ sets of vertices, $V_1, V_2, \ldots, V_{t-1}$, each of size t^ϵ. If a hyperedge contains vertices from three different V_i, then color it green. By the well-known lower bound on the diagonal Ramsey number (Theorem 1.34) $R(K_{t^{1-\epsilon}}, K_{t^{1-\epsilon}}) = \Omega(2^{t^{1-\epsilon}/2})$, we can find a coloring of K_{t-1} containing no clique of size $t^{1-\epsilon}$ when t is sufficiently large. Given such a red-blue coloring on the complete graph with vertex set $\{1, 2, \ldots, t-1\}$ we color the hyperedges consisting of two vertices from V_i and one from V_j by the color of $\{i, j\}$ in the graph. We color every hyperedge completely contained in some V_i red. Observe that the core of any red or blue BK_t may contain vertices in less than $t^{1-\epsilon}$ different classes and so has a total of less than t vertices. □

Theorem 5.38 (Salia, Tompkins, Wang, Zamora. [79])**.** *For any uniformity* $r \geq 4$*, and sufficiently large* c *and* t*, we have*

$$R^r_c(BK_t, BK_t, \ldots, BK_t) > t^{1+\left(\frac{r-3}{r-2}\right)^{r-3} - \left(\frac{r-3}{r-2}\right)^{r-2}}.$$

Theorem 5.38 will follow from the following claim which we will prove by induction on r by choosing the optimal ϵ.

Claim 5.39. *For any uniformity* $r \geq 3$*, and for any* ϵ *where* $\epsilon < 1$*, for sufficiently large* c *and* t*, we have*

$$R^r_c(BK_t, BK_t, \ldots, BK_t) > t^{1+(1-\epsilon)^{r-3} - (1-\epsilon)^{r-2}}.$$

Proof. The base case follows from Claim 5.37. Now assume that $r \geq 4$. Let $\epsilon < 1$. Let c_s be the number of colors required for Claim 5.39 to hold for an s-uniform hypergraph for $2 \leq s \leq r-1$. Let M be the lower bound we obtain by induction for the function $R^{r-1}_{c_{r-1}}(BK_{t^{1-\epsilon}}, BK_{t^{1-\epsilon}}, \ldots, BK_{t^{1-\epsilon}})$. We will show

$$R^r_{c_r}(BK_t, BK_t, \ldots, BK_t) > M \cdot t^\epsilon.$$

for some constant c_r depending on r.

Take the complete r-uniform hypergraph $\mathcal{H}$ on $N = M \cdot t^\epsilon$ vertices. Partition the vertex set into sets $V_1, V_2, \ldots, V_M$ each consisting of t^ϵ vertices. We consider s-uniform complete hypergraphs $\mathcal{H}_s$ defined on the vertex set $\{1, 2, \ldots, M\}$ for $2 \leq s \leq r-1$. Since the lower bounds in Claim 5.39 are decreasing (in r), we have for c_s colors a coloring of $\mathcal{H}_s$ with no Berge clique of size $t^{1-\epsilon}$ provided t is sufficiently large. Assume, indeed, that t is at least the maximum required for any s.

Now, given the colorings of $\mathcal{H}_i$ with c_i colors, for $2 \leq i \leq r-1$, we define a coloring on $\mathcal{H}$ with $c_r = \sum_{s=2}^{r-1} c_s + 2$ colors and no monochromatic BK_t. For $2 \leq s \leq r-1$ we color all hyperedges containing elements of the vertex sets $V_{i_1}, V_{i_2}, \ldots, V_{i_s}$ with the same color as $\{i_1, i_2, \ldots, i_s\}$ in the coloring of $\mathcal{H}_s$. Observe that the core of a monochromatic BK_t in $\mathcal{H}$ can contain vertices from fewer than $t^{1-\epsilon}$ classes. Since $\mathcal{H}_s$ has no monochromatic $BK_{t^{1-\epsilon}}$, and each class has t^ϵ vertices, it follows that $\mathcal{H}$ has no monochromatic BK_t using hyperedges containing vertices from between 2 and $r-1$ classes. Finally, we may color the hyperedges contained in each V_i with any color used so far and the hyperedges containing vertices from r classes with a new color.

It remains to verify that $M \cdot t^\epsilon$ yields the required bound. Indeed,

$$M \cdot t^\epsilon = t^{(1-\epsilon)\left(1+(1-\epsilon)^{r-4}-(1-\epsilon)^{r-3}\right)} \cdot t^\epsilon = t^{1+(1-\epsilon)^{r-3}-(1-\epsilon)^{r-2}}. \qquad \square$$

We now discuss briefly the case of forbidding Berge-cliques of higher uniformity. First we collect some basic lemmas about the Ramsey number for Berge cliques in different uniformities.

Lemma 5.40. *For any r, c, a, b, where $a < b$ and for t sufficiently large, we have*

$$R_c^r(BK_t^{(b)}, BK_t^{(b)}, \ldots, BK_t^{(b)}) \geq R_c^r(BK_t^{(a)}, BK_t^{(a)}, \ldots, BK_t^{(a)}).$$

Proof. It is sufficient to prove that for sufficiently large t, there is an injection from $\binom{[t]}{a}$ to $\binom{[t]}{b}$ mapping sets to one of their supersets. Let $S \subset \binom{[t]}{a}$ and $\phi(S)$ be the elements of $\binom{[t]}{b}$ which contain some element from S. We have $|S|\binom{t-a}{b-a} \leq |\phi(S)|\binom{b}{a}$ by double-counting the relations between the two levels. Then $|\phi(S)| \geq |S|$ is obvious for sufficiently large t, and we have the desired injection by Hall's theorem. $\square$

Corollary 5.41. *For any uniformity r, $a < r$, and sufficiently large c and t, we have*

$$R_c^r(BK_t^{(a)}, BK_t^{(a)}, \ldots, BK_t^{(a)}) \geq t^{1+\left(\frac{r-3}{r-2}\right)^{r-3}-\left(\frac{r-3}{r-2}\right)^{r-2}}.$$

Proof. The result is immediate from Lemma 5.40 and Theorem 5.38. $\square$

5.3 Ramsey numbers of 2-shadow graphs and proof of Theorem 5.14

In this short section, we discuss some results on the Ramsey number of $R^r(\partial K_t, \partial K_s)$. On the one hand, we have $R^r(\partial K_t, \partial K_s) \leq R^r(BK_t, BK_s)$. Most of the constructions from Section 5.2 are also constructions for $R^r(\partial K_t, \partial K_s)$; however, there are some exceptions.

Proposition 5.42. *Let $s, t \geq 3$, we have $R^3(\partial K_2, \partial K_2) = 3$, $R^3(\partial K_2, \partial K_s) = s$ and $R^3(\partial K_t, \partial K_s) = t + s - 3$.*

Proof. It is easy to see that $R^3(\partial K_2, \partial K_2) = 3$ and $R^3(\partial K_2, \partial K_s) = s$ for $s \geq 3$. We will now show $R^3(\partial K_t, \partial K_s) \leq t + s - 3$ for $s, t \geq 3$ by inducting on $s + t$. The cases when s or t is 3 are trivial. Assume the theorem holds for smaller $s + t$ and take a 2-edge-colored complete 3-uniform hypergraph $\mathcal{H}$ on the vertex set V of size $s + t - 3$ where $s, t \geq 4$. If for all $x, y \in V$ we have that there exists z such that $\{x, y, z\}$ is blue, then we have complete blue clique in the 2-shadow. Otherwise suppose there is a pair of vertices x, y such that for all $z \in V \setminus \{x, y\}$ we have $\{x, y, z\}$ is red, then consider the subhypergraph of $\mathcal{H}$ induced by $V \setminus \{x\}$. By induction, there exists either a blue ∂K_t, in which case we are done, or a red ∂K_{s-1} with Y as its core. Then we can extend it to a red ∂K_s with $Y \cup \{x\}$ as its core by adding the red hyperedges $\{x, y, z\}$ where $z \in Y$.

The lower bound construction is to take a set of $t - 2$ vertices A and a set of $s - 2$ vertices B and color a hyperedge red if and only if it intersects A in at most 1 vertex. $\square$

Proposition 5.43. *For $r \geq 4$ and $s, t \geq 2$, we have $R^r(\partial K_t, \partial K_s) = \max\{s, t, r\}$.*

Proof. Consider a 2-edge-colored complete r-uniform hypergraph on $N = \max\{s, t, r\}$ vertices. Suppose first, that for every pair $x, y \in V$ there exists $z_1, z_2, \ldots, z_{r-2}$ such that $\{x, y, z_1, z_2, \ldots, z_{r-2}\}$ is blue, then there is a blue K_N in the shadow. On the other hand, if there are $x, y \in V$, such that for all $z_1, z_2, \ldots, z_{r-2}$, $\{x, y, z_1, z_2, \ldots, z_{r-2}\}$ is red, then it is easy to see that there is a red K_N in the 2-shadow. Thus, $R^r(\partial K_t, \partial K_s) \leq \max\{s, t, r\}$. On the other hand taking a clique of the appropriate color on $\max\{s, t, r\} - 1$ vertices yields a construction for the lower bound. $\square$

Remark 5.44. *The superlinear lower bounds constructed in Subsection 5.2.4 are in fact constructions for hypergraphs without monochromatic cliques in the 2-shadow. Thus, the same lower bounds hold.*

5.4 Ramsey numbers of trace-cliques

Throughout this section, we assume that a, b are positive integers.

Lemma 5.45. *Let $t \geq a + 1, s \geq b + 1$. Then*

$$R^{a+b+1}(TK_t^{(a+1)}, TK_s^{(b+1)}) \leq R^{a+b+1}(TK_{t-1}^{(a+1)}, TK_s^{(b+1)}) + s - b.$$

Proof. Let $N = R^{a+b+1}(TK_{t-1}^{(a+1)}, TK_s^{(b+1)}) + s - b$, and $\mathcal{H}$ be a 2-edge-colored (blue and red) complete $(a + b + 1)$-uniform hypergraph on N vertices. Let $\mathcal{H}'$ be an induced subhypergraph of $\mathcal{H}$ on $R^{a+b+1}(TK_{t-1}^{(a+1)}, TK_s^{(b+1)}) = N - (s - b)$ vertices, obtained by removing a set Y of $s - b$ vertices. Then $\mathcal{H}'$ contains either a blue $TK_{t-1}^{(a+1)}$ or a red $TK_s^{(b+1)}$. In the second case we are done, so let us assume that $\mathcal{H}'$ contains a blue $TK_{t-1}^{(a+1)}$ with core X. Let Z be a set of b vertices of $\mathcal{H}'$ which does not intersect X (there is such a set since $v(\mathcal{H}') \geq v(TK_{t-1}^{(a+1)}) \geq t - 1 + b$) and put $S = Y \cup Z$. Consider the edges of the form $A \cup B$ where $A \subseteq X, |X| = a$ and $B \subseteq S, |B| = b + 1$. If for some fixed B, $A \cup B$ is blue for every subset A of X of size a, then pick $v \in B \cap Y$, and together with these edges and the edges defining the blue $TK_{t-1}^{(a+1)}$, $X \cup \{v\}$ is the core of a blue $TK_{t-1}^{(a+1)}$. If this is not the case, then for any $B \subset S$ of size $b + 1$, there exists $A_B \subseteq S$ such that $A_B \cup B$ is red, and therefore, S together with these edges is the core of a red $TK_s^{(b+1)}$. $\square$

Theorem 5.46 (Salia, Tompkins, Wang, Zamora. [79])**.** *Let $t \geq a+1, s \geq b+1$. Then*

$$R^{a+b+1}(TK_t^{(a+1)}, TK_s^{(b+1)}) \leq (t-a)(s-b) + a + b.$$

Proof. We are going to prove this result by induction on t, the base case is where $t = a+1$, we have that $R^{a+b+1}(TK_{a+1}^{(a+1)}, TK_s^{(b+1)}) = s + a = (s-b) + b + a$, so the result follows. Now assume that for some $t - 1 \geq a+1$ the result is true, then by Lemma 5.45 we have

$$\begin{aligned} R^{a+b+1}(TK_t^{(a+1)}, TK_s^{(b+1)}) &\leq R^{a+b+1}(TK_{t-1}^{(a+1)}, TK_s^{(b+1)}) + s - b \\ &\leq (t-1-a)(s-b) + a + b + s - b \\ &= (t-a)(s-b) + a + b. \end{aligned}$$

□

Proposition 5.47. *Suppose that $t \geq a+1 \geq 3$ and $s \geq 2$. Then*

$$R^{a+1}(SK_t^{(a)}, TK_s) \leq t + \max\{R^{a+1}(SK_{t-1}^{(a)}, TK_s), R^{a+1}(SK_t^{(a)}, TK_{s-1})\}.$$

Proof. Let $\mathcal{H}$ be an $(a+1)$-uniform hypergraph with vertex set V of size

$$N = t + \max\{R^{a+1}(SK_{t-1}^{(a)}, TK_s), R^{a+1}(SK_t^{(a)}, TK_{s-1})\}.$$

Since $N > R^{a+1}(SK_{t-1}^{(a)}, TK_s)$, it follows that we can find either a blue $SK_{t-1}^{(a)}$ or a red TK_s. In the latter case, we are done, so assume there is a blue $SK_{t-1}^{(a)}$ with defining vertices X and suspension vertex u. Now, if for some $v \in V \backslash (X \cup \{u\})$ it holds that for every set $A \subseteq X$ of size $a-1$ we have that $A \cup \{v, u\}$ is blue, then we can add v to X and obtain a blue $SK_t^{(a)}$. Otherwise suppose that for every v we can find a set A_v such that $A_v \cup \{v, u\}$ is red. Let $V' = V \setminus (X \cup \{u\})$. Note that $|V'| \geq R^{a+1}(SK_t^{(a)}, TK_{s-1})\}$. It follows that we can find either a blue $SK_t^{(a)}$ or a red TK_{s-1} in $\mathcal{H}[V']$. If we find a blue $SK_t^{(a)}$, we are done. Otherwise suppose we can find a red TK_{s-1} defined on the set Y. Then we can extend Y to a red TK_s by adding to Y the vertex u together with the edges $A_v \cup \{v, u\}$ for every $v \in Y$ since A_v does not intersect V'. □

Corollary 5.48. *Suppose that $t \geq a \geq 2$ and $s \geq 2$. Then*

$$R^{a+1}(SK_t^{(a)}, TK_s) \leq \binom{t}{2} + (s-1)t.$$

Proof. This bound follows by induction on $s+t$ from Proposition 5.47. The case when $s = 2$ or $t = a$ are trivial. Assume we had the bound for smaller values of $s+t$ and observe that Proposition 5.47 and induction imply that $R^{a+1}(SK_t^{(a)}, TK_s)$ is bounded by

$$t + \max\left(\binom{t-1}{2} + (s-1)(t-1), \binom{t}{2} + (s-2)t\right) \leq \binom{t}{2} + (s-1)t,$$

as required. □

Proposition 5.49. *Suppose that $t \geq a+1$ and $s \geq 2$. Then*

$$R^{a+1}(SK_t^{(a)}, \partial K_s) \geq (s-1)\left\lfloor \frac{t}{a} \right\rfloor + 1.$$

Proof. Take a vertex set of size $(s-1)\left\lfloor\frac{t}{a}\right\rfloor$ and divide it into $s-1$ classes $V_1, V_2, \ldots, V_{s-1}$ of size at most $\left\lfloor\frac{t}{a}\right\rfloor$. Color every hyperedge which intersects each V_i in at most 1 with red, and color every other hyperedge blue. Clearly this construction has no red ∂K_s. We will now show it has no blue $SK_t^{(a)}$. Indeed, suppose that X is the core of the blue suspension and v is the suspension vertex.

Let $V_{i_1}, \ldots, V_{i_k}$ denote the classes which have nonempty intersection with $X \cup \{v\}$, then $t+1 = |X \cup \{v\}| = \sum_{j=1}^{k} \left|(X \cup \{v\}) \cap V_{i_j}\right| \leq \frac{kt}{a}$. It follows that $k > a$. Suppose, without loss of generality, that $v \in V_{i_{a+1}}$. Then we may take $x_j \in X \cap V_{i_j}$ for $j = 1, \ldots, a$ so that the edge $\{x_1, \ldots, x_a, v\}$ is red, and thus not a member of a blue suspension, contradiction. □

Thus, we have the following corollaries.

Corollary 5.50. *Suppose that $t \geq a+1$ and $s \geq 2$. Then*

$$R^{a+1}(SK_t^{(a)}, TK_s) \geq (s-1)\left\lfloor\frac{t}{a}\right\rfloor + 1.$$

Corollary 5.51. $R^{a+1}(SK_t^{(a)}, TK_t) = \Theta_a(t^2)$.

Proposition 5.52. *Suppose that $t \geq a+2$ and $s \geq b+2$. Then*

$$R^{a+b+1}(HK_t^{(a+1)}, TK_s^{(b+1)}) \leq M + t + b\binom{t}{a+1} - b,$$

where $M = \max\left(R^{a+b+1}(HK_{t-1}^{(a+1)}, TK_s^{(b+1)}), R^{a+b+1}(HK_t^{(a+1)}, TK_{s-1}^{(b+1)})\right)$.

Proof. Let $\mathcal{H}$ be an $(a+b+1)$-uniform hypergraph with vertex set V of size

$$N = M + t + b\binom{t}{a+1} - b.$$

Since $N > M$, we can find either a blue $HK_{t-1}^{(a+1)}$ or a red $TK_s^{(b+1)}$. If the latter case occurs we are done, so assume there is a blue $HK_{t-1}^{(a+1)}$ with core X of size $t-1$ and set of expansion vertices X' of size $\binom{t-1}{a+1}b$. Now let v be a vertex not in $X \cup X'$. We will try to extend X together with v. Let $A_1, A_2, \ldots, A_{\binom{t-1}{a}}$ be an ordering of the subsets of X of size a. Let $V_1 = V \backslash (X \cup X' \cup \{v\})$ and set $X_1 = X'$. For each $i = 1, 2 \ldots, \binom{t-1}{a}$, if there is a set B_i of size b in V_i such that that $B_i \cup A_i \cup \{v\}$ is blue, then set $V_{i+1} = V_i \backslash B_i$ and $X_{i+1} = X_i \cup B_i$, otherwise we stop. If we can do this for every i then the set $X \cup \{v\}$ defines a blue $HK_t^{(a+1)}$ with expansion set $X_{\binom{t-1}{a}}$. If not, then there is an index i such that we have to stop. This means that for every set B of size b in V_i we have that $A_i \cup B \cup \{v\}$ is red. Now the size of V_i is $N - (t-1) - \binom{t-1}{a+1}b - (i-1)b - 1 \geq N - t - \binom{t-1}{a+1}b - (\binom{t-1}{a} - 1)b = M$. So by the definition of M, we can find either a blue $HK_t^{(a+1)}$ using V_i or a red $TK_{s-1}^{(b+1)}$. In the first case we are done, so suppose we have a red $TK_{s-1}^{(b+1)}$ with defining vertices Y. Now we can extend Y together with v to a red $TK_s^{(b+1)}$, since for every $B \subseteq Y$ of size b we have that the edge $B \cup A_i \cup \{v\}$ is red. □

Corollary 5.53. *Suppose that $t \geq a+1$ and $s \geq b+1$. Then*

$$R^{a+b+1}(HK_t^{(a+1)}, TK_s^{(b+1)}) \leq b\binom{t+1}{a+2} + \binom{t+1}{2} - tb + s\left(b\binom{t}{a+1} + t - b\right).$$

5.5 Ramsey number of expansion and suspension hypergraphs

5.5.1 Expansion hypergraphs and Proof of Theorem 5.9

In this section, we give an upper bound on $R^3(HK_t, HK_s)$. Recall that $H^r(K_t)$ is the the r-graph obtained from the complete graph K_t by enlarging each edge by a set of $(r-2)$ distict new vertices. Moreover, $R^r(H^r(K_t), H^r(K_t))$ is the smallest integer n such that every 2-edge-coloring of the complete r-uniform hypergraph $\mathcal{H}$ on n vertices contains a monochromatic $H^r(K_t)$. For ease of reference, we will use $R^r(HK_t, HK_t)$ to denote $R^r(H^r(K_t), H^r(K_t))$. We first prove the following lemma.

Lemma 5.54. *For $s, t \geq 2$, we have that*

$$R^3(HK_{t+1}, HK_{s+1}) \leq \max\{R^3(HK_{t+1}, HK_s), R^3(HK_t, HK_{s+1})\} + 2st.$$

Proof. Without loss of generality, we assume that $t \leq s$. Let

$$N = \max\{R^3(HK_{t+1}, HK_s), R^3(HK_t, HK_{s+1})\} + 2st$$

and $\mathcal{H}_N$ be a 2-edge-colored compete 3-uniform hypergraph on N vertices. Let

$$W = \{v_1, v_2, \ldots, v_{2st}\} \subset V(\mathcal{H}_N)$$

and $\mathcal{H}' = \mathcal{H}[V(\mathcal{H}_N)\backslash W]$.

Note that $|\mathcal{H}'| \geq R^3(HK_t, HK_{s+1})$. Thus by definition of Ramsey number, there exists either a blue expansion of K_t or a red expansion of K_{s+1}. If the latter happens, we are done. Thus, assume that we have a blue expansion $\mathcal{H}_b$ of K_t. Note that $\mathcal{H}_b$ has $\binom{t}{2} + t$ vertices. Let $\{u_1, \ldots u_t\}$ be the core of $\mathcal{H}_b$. Let $F = V(\mathcal{H})\backslash V(\mathcal{H}_b)$.

Claim 5.55. *Suppose that $\mathcal{H}_N$ does not have a blue expansion of K_{t+1}. Then for every $v \in W$, there exists some u in the core of $\mathcal{H}_b$ such that $\{v, u, w\}$ is colored red for all w except at most $(t-1)$ elements from $F\backslash\{v\}$.*

Proof. Fix a vertex $v \in W$. Construct a bipartite graph $G = A \cup B$ where $A = \{u_1, \ldots, u_t\}$ and $B = F\backslash\{v\}$. For $u_i \in A$, $w \in B$, u_i is adjacent to w in G if and only if $\{v, u_i, w\}$ is a blue edge in $\mathcal{H}_N$. Note that for every $w \in B$, $d_G(w) \leq t$. Therefore, if $d_G(u_i) \geq t$ for every $u_i \in A$, then there exists a matching of A in G by Hall's theorem, which implies that we can extend $\mathcal{H}_b$ to a blue expansion of K_{t+1} by adding v into the core of $\mathcal{H}_b$. This contradicts our assumption that $\mathcal{H}_N$ does not have a blue expansion of K_{t+1}. Hence it follows that there exists a vertex $v' \in A$ such that $\{v, v', w\}$ is colored red for all except $t-1$ elements of $F\backslash\{v\}$. This finishes the proof of Claim 5.55 □

Now since $|W| = 2st$, by pigeonhole principle, there exists some u in the core of $\mathcal{H}_b$ so that there exists $W_u = \{w_1, w_2, \ldots, w_s\}$ such that for any $w \in W_u$, the hyperedge $\{w, u, w'\}$ is red for all w' except at most $(t-1)$ elements of $F\backslash\{w\}$. Let $M(w_i)$ be the elements w' in W such that $\{u, w_i, w'\}$ is blue.

Now let $W' = W_u \cup V(\mathcal{H}_b) \cup \bigcup_{i=1}^{s} M(w_i)$ and $\mathcal{H}'' = \mathcal{H}_N[V(\mathcal{H}_N)\backslash W']$. Note that $|\mathcal{H}''| \geq R^3(HK_{t+1}, HK_s)$ since $2st \geq st + \binom{t}{2} + t$. Hence there either exists a blue expansion of K_{t+1} or there exists a red expansion of K_s. If the former happens, we are done. Hence assume we have a red expansion $\mathcal{H}_r$ of K_s. Suppose $\{v_1, v_2, \ldots, v_s\}$ is the core of $\mathcal{H}_r$. Now we can extend $\mathcal{H}_r$ to a red expansion of K_{s+1} by adding u into the core of $\mathcal{H}_r$ together with the red edges in $\{\{u, w_i, v_i\} : i \in [s]\}$. This completes the proof of the lemma. □

Now we are ready to show that $R^3(HK_t, HK_s) \leq 2(s+t)st$. The proof is by induction on $s+t$. We first show that $R^3(HK_2, HK_s) \leq 4s^2+8s$. This is clearly true since any blue edge in a 3-uniform hypergraph is a blue expansion of K_2. Hence given any 2-edge-colored complete 3-uniform hypergraph $\mathcal{H}$ with $4s^2+8s$ vertices, if there is no blue edge, then all edges are red, which implies that we have a red expansion of K_s, since $4s^2+8s \geq \binom{s}{2}+s$. Similarly, $R^3(HK_t, HK_2) \leq 4t^2+8t$.

Now assume the theorem holds for $HK_{t'}, HK_{s'}$ such that $t'+s' < t+s$. Without loss of generality, assume that $t \leq s$. Then by the Lemma 5.54,

$$\begin{aligned} R^3(HK_t, HK_s) &\leq \max\{R^3(HK_t, HK_{s-1}), R^3(HK_{t-1}, HK_s)\} + 2(s-1)(t-1) \\ &\leq 2(s+t-1)t(s-1) + 2(s-1)(t-1) \\ &\leq 2st(s+t). \end{aligned}$$

Hence we are done by induction.

5.5.2 Ramsey number of suspension hypergraphs

Recall that the r-suspension SK_t of the complete graph K_t, is the r-uniform hypergraph formed by adding a single fixed set of $r-2$ distinct new vertices to every edge in K_t. Clearly, $R^r(SK_t, SK_t) \leq R^2(K_t, K_t) + (r-2)$. The proof is simple: let $\mathcal{H}$ be a 2-edge-colored $K^{(r)}_{R^2(K_t,K_t)+(r-2)}$. Fix a set of $(r-2)$ vertices S and consider the complete graph G on the remaining $R^2(K_t, K_t)$ vertices, where the color of an edge e in G is the same color as the hyperedge $e \cup S$ in $\mathcal{H}$. By the definition of the Ramsey number, there exists a monochromatic clique in G, which gives us the core of the monochromatic SK_t in $\mathcal{H}$.

Before we prove the lower bound, let us recall the symmetric version of the Lovász local lemma [4]:

Lemma 5.56 (Lovász [4])**.** *Let $\mathcal{A} = \{A_1, \ldots, A_q\}$ be a finite set of events in the probability space Ω. Suppose that each event A_i is mutually independent of a set of all but at most d of the other events A_j, and that* $\Pr(A_i) \leq p$ *for all $1 \leq i \leq q$. If*

$$ep(d+1) < 1,$$

then

$$\Pr\left(\bigwedge_{i=1}^{q} \overline{A_i}\right) > 0.$$

Now we can show a lower bound of $R^r(SK_t, SK_t)$ with the local lemma.

Proposition 5.57. *Fix $t \geq r \geq 3$. If*

$$e\left(1 + \binom{t}{2}\binom{r}{2}\binom{n}{t-2}\right)2^{1-\binom{t}{2}} < 1,$$

then $R^r(SK_t, SK_t) > n$.

Proof. Let $\mathcal{H}$ be a complete r-uniform hypergraph on n vertices. Color each hyperedge blue or red randomly and independently with probability $\frac{1}{2}$. For a set of $r-2$ vertices S and another set of t vertices T disjoint from S, let $A_{S,T}$ be the event that the suspension hypergraph $\mathcal{H}_{S,T}$ with T as core and S as the suspending vertex set is monochromatic. Note that for each fixed S, T,

$$\Pr(A_{S,T}) = 2^{1-\binom{t}{2}} = p.$$

Note that $A_{S,T}$ is mutually independent of all other events $A_{S',T'}$ satisfying $E(\mathcal{H}_{S,T}) \cap E(\mathcal{H}_{S',T'}) = \emptyset$. Let us give an upper bound on the number of events $A_{S',T'}$ that $A_{S,T}$ is mutually dependent of. There are $\binom{t}{2}$ choices to pick an edge they share, which contains r vertices. Among the r vertices, $r-2$ of them must be the suspension vertices. There are $\binom{r}{r-2}$ ways to choose the suspension vertices S'. There are then at most $\binom{n}{t-2}$ ways to choose the remaining $t-2$ vertices of T. Hence it follows that

$$d \leq \binom{t}{2}\binom{r}{2}\binom{n}{t-2}.$$

By the Lovász local lemma, it follows then that if $ep(d+1) < 1$, we have that

$$\Pr\left(\bigwedge_{S,T} \overline{A_{S,T}}\right) > 0.$$

Hence there exists a coloring of $\mathcal{H}$ without any monochromatic SK_t. $\square$

Remark 5.58. *For any fixed r, this gives asymptotically the same lower bound as Ramsey number $R^2(K_t, K_t)$, i.e. $R^r(SK_t, SK_t) > (1+o(1))\frac{\sqrt{2}}{e}t\sqrt{2}^t$.*

Bibliography

[1] N. Alon. On the number of subgraphs of prescribed type of graphs with a given number of edges. *Israel Journal of Mathematics*, 38(1-2):116–130, 1981.

[2] N. Alon and Y. Caro. On the number of subgraphs of prescribed type of planar graphs with a given number of vertices. In *North-Holland Mathematics Studies*, volume 87, pages 25–36. Elsevier, 1984.

[3] N. Alon and C. Shikhelman. Many T copies in H-free graphs. *Journal of Combinatorial Theory, Series B*, 121:146–172, 2016.

[4] N. Alon and J. H. Spencer. *The Probabilistic Method*. John Wiley & Sons, 2016.

[5] R. Anstee and S. Salazar. Forbidden Berge Hypergraphs. *The Electronic Journal of Combinatorics*, 24(1):P1–59, 2017.

[6] M. Axenovich and A. Gyárfás. A note on Ramsey numbers for Berge-G hypergraphs. *Discrete Mathematics*, 342(5):1245–1252, 2019.

[7] P. N. Balister, E. Győri, J. Lehel, and R. H. Schelp. Connected graphs without long paths. *Discrete Mathematics*, 308(19):4487–4494, 2008.

[8] C. Berge. *Graphs and hypergraphs*. North-Holland Pub. Co., 1973.

[9] B. Bollobás and E. Győri. Pentagons vs. triangles. *Discrete Mathematics*, 308(19):4332–4336, 2008.

[10] J. A. Bondy and M. Simonovits. Cycles of even length in graphs. *Journal of Combinatorial Theory, Series B*, 16(2):97–105, 1974.

[11] V. Chvátal. Tree-complete graph Ramsey numbers. *Journal of Graph Theory*, 1(1):93–93, 1977.

[12] D. Conlon, J. Fox, and V. Rödl. Hedgehogs are not colour blind. *Journal of Combinatorics*, 8(3):475–485, 2017.

[13] D. Conlon, J. Fox, and B. Sudakov. Hypergraph Ramsey numbers. *Journal of the American Mathematical Society*, 23(1):247–266, 2010.

[14] A. Davoodi, E. Győri, A. Methuku, and C. Tompkins. An Erdős–Gallai type theorem for uniform hypergraphs. *European Journal of Combinatorics*, 69:159–162, 2018.

[15] G. A. Dirac. Some theorems on abstract graphs. *Proceedings of the London Mathematical Society*, 3(1):69–81, 1952.

[16] C. Dowden. Extremal C_4-Free/C_5-Free Planar Graphs. *Journal of Graph Theory*, 83(3):213–230, 2016.

[17] D. Eppstein. Connectivity, graph minors, and subgraph multiplicity. *Journal of Graph Theory*, 17(3):409–416, 1993.

[18] P. Erdős. On the number of complete subgraphs contained in certain graphs. *Magyar Tud. Akad. Mat. Kutató Int. Közl*, 7(3):459–464, 1962.

[19] P. Erdős. On some problems in graph theory, combinatorial analysis and combinatorial number theory. *Graph Theory and Combinatorics (Cambridge, 1983), Academic Press, London*, pages 1–17, 1984.

[20] P. Erdös. Some remarks on the theory of graphs. *Bulletin of the American Mathematical Society*, 53(4):292–294, 1947.

[21] P. Erdös. Graph theory and probability. II. *Canadian Journal of Mathematics*, 13:346–352, 1961.

[22] P. Erdős. Extremal problems in graph theory, Theory of graphs and its applications. In *Proc. of the Symposium held in Smolenice in June*, pages 29–38, 1963.

[23] P. Erdős and T. Gallai. On maximal paths and circuits of graphs. *Acta Mathematica Hungarica*, 10(3-4):337–356, 1959.

[24] P. Erdős and M. Simonovits. A limit theorem in graph theory. In *Studia Sci. Math. Hung.* Citeseer, 1965.

[25] P. Erdős and A. H. Stone. On the structure of linear graphs. *Bull. Amer. Math. Soc*, 52(1087-1091):1, 1946.

[26] P. Erdös and G. Szekeres. A combinatorial problem in geometry. *Compositio Mathematica*, 2:463–470, 1935.

[27] R. Fagin. Degrees of acyclicity for hypergraphs and relational database schemes. *Journal of the ACM (JACM)*, 30(3):514–550, 1983.

[28] J. Fox and R. Li. On Ramsey numbers of hedgehogs. *Combinatorics, Probability and Computing*, 29(1):101–112, 2020.

[29] J. Fox and F. Wei. On the number of cliques in graphs with a forbidden minor. *Journal of Combinatorial Theory, Series B*, 126:175–197, 2017.

[30] P. Frankl and Z. Füredi. Exact solution of some Turán-type problems. *Journal of Combinatorial Theory, Series A*, 45(2):226–262, 1987.

[31] P. Frankl and J. Pach. On disjointly representable sets. *Combinatorica*, 4(1):39–45, 1984.

[32] Z. Füredi. Linear trees in uniform hypergraphs. *European Journal of Combinatorics*, 35:264–272, 2014.

[33] Z. Füredi and J. Pach. Traces of finite sets: extremal problems and geometric applications. *Extremal Problems for Finite Sets*, 3:255–282, 1991.

[34] D. Gerbner. A note on the Turán number of a Berge odd cycle. *arXiv preprint arXiv:1903.01002*, 2019.

[35] D. Gerbner, E. Győri, A. Methuku, and M. Vizer. Generalized Turán problems for even cycles. *arXiv preprint arXiv:1712.07079*, 2017.

[36] D. Gerbner, A. Methuku, G. Omidi, and M. Vizer. Ramsey problems for Berge hypergraphs. *SIAM Journal on Discrete Mathematics*, 34(1):351–369, 2020.

[37] D. Gerbner, A. Methuku, and C. Palmer. General lemmas for Berge-Turán hypergraph problems. *arXiv preprint arXiv:1808.10842*, 2018.

[38] D. Gerbner and C. Palmer. Extremal results for Berge hypergraphs. *SIAM Journal on Discrete Mathematics*, 31(4):2314–2327, 2017.

[39] L. Gishboliner and A. Shapira. A generalized Turán problem and its applications. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, pages 760–772. ACM, 2018.

[40] D. Grósz, A. Methuku, and C. Tompkins. Uniformity thresholds for the asymptotic size of extremal Berge-F-free hypergraphs. *Electronic Notes in Discrete Mathematics*, 61:527–533, 2017.

[41] A. Grzesik. On the maximum number of five-cycles in a triangle-free graph. *Journal of Combinatorial Theory, Series B*, 102(5):1061–1066, 2012.

[42] A. Gyárfás. The Turán Number of Berge K_4 in Triple Systems. *SIAM Journal on Discrete Mathematics*, 33(1):383–392, 2019.

[43] A. Gyárfás, J. Lehel, G. N. Sárközy, and R. H. Schelp. Monochromatic Hamiltonian Berge-cycles in colored complete uniform hypergraphs. *Journal of Combinatorial Theory, Series B*, 98(2):342–358, 2008.

[44] A. Gyárfás and G. N. Sárközy. The 3-colour Ramsey number of a 3-uniform Berge cycle. *Combinatorics, Probability and Computing*, 20(1):53–71, 2011.

[45] E. Győri. On the number of C_5's in a triangle-free graph. *Combinatorica (Print)*, 9(1):101–102, 1989.

[46] E. Győri. Triangle-free hypergraphs. *Combinatorics, Probability and Computing*, 15(1-2):185–191, 2006.

[47] E. Győri, G. Y. Katona, and N. Lemons. Hypergraph extensions of the Erdős-Gallai theorem. *European Journal of Combinatorics*, 58:238–246, 2016.

[48] E. Győri, N. Lemons, N. Salia, and O. Zamora. The Structure of Hypergraphs without long Berge cycles. *arXiv preprint arXiv:1812.10737*, 2018.

[49] E. Győri and H. Li. The maximum number of triangles in C_{2k+1}-free graphs. *Combinatorics, Probability and Computing*, 21(1-2):187–191, 2012.

[50] E. Győri, A. Paulos, N. Salia, C. Tompkins, and O. Zamora. The Maximum Number of Pentagons in a Planar Graph. *arXiv preprint arXiv:1909.13532*, 2019.

[51] E. Győri, N. Salia, C. Tompkins, and O. Zamora. The maximum number of P_ℓ copies in P_k-free graphs. *Discrete Mathematics and Theoretical Computer Science*, 21(11):#14, 2019.

[52] E. Győri, N. Salia, C. Tompkins, and O. Zamora. Turán numbers of Berge trees. *arXiv preprint arXiv:1904.06728*, 2019.

[53] S. L. Hakimi and E. F. Schmeichel. On the number of cycles of length k in a maximal planar graph. *Journal of Graph Theory*, 3(1):69–86, 1979.

[54] P. Hall. On representatives of subsets. In *Classic Papers in Combinatorics*, pages 58–62. Springer, 2009.

[55] H. Hatami, J. Hladkỳ, D. Král, S. Norine, and A. Razborov. On the number of pentagons in triangle-free graphs. *Journal of Combinatorial Theory, Series A*, 120(3):722–732, 2013.

[56] P. E. Haxell, T. Łuczak, Y. Peng, V. Rödl, A. Ruciński, M. Simonovits, and J. Skokan. The Ramsey number for hypergraph cycles I. *Journal of Combinatorial Theory, Series A*, 113(1):67–83, 2006.

[57] J. T. Johnston and L. Lu. Turán Problems on Non-Uniform Hypergraphs. *The Electronic Journal of Combinatorics*, 21(4):P4–22, 2014.

[58] P. Keevash. Hypergraph Turán problems. *Surveys in Combinatorics*, 392:83–140, 2011.

[59] G. Kopylov. On maximal paths and cycles in a graph. *Russian Academy of Sciences*, 234(1):19–21, 1977.

[60] T. Kővári, V. Sós, and P. Turán. On a problem of K. Zarankiewicz. In *Colloquium Mathematicum 3*, pages 50–57, 1954.

[61] C. Kuratowski. Sur le probléme des courbes gauches en topologie. *Fundamenta mathematicae*, 15(1):271–283, 1930.

[62] Y. Lan and Y. Shi. Planar Turán Numbers of Short Paths. *Graphs and Combinatorics*, 35(5):1035–1049, 2019.

[63] Y. Lan, Y. Shi, and Z.-X. Song. Extremal H-free planar graphs. *arXiv preprint arXiv:1808.01487*, 2018.

[64] Y. Lan, Y. Shi, and Z.-X. Song. Extremal theta-free planar graphs. *Discrete Mathematics*, 342(12):111610, 2019.

[65] C. Lee and S. Oum. Number of cliques in graphs with a forbidden subdivision. *SIAM Journal on Discrete Mathematics*, 29(4):1999–2005, 2015.

[66] S. Letzter. Many H-copies in graphs with a forbidden tree. *SIAM Journal on Discrete Mathematics*, 33(4):2360–2368, 2019.

[67] R. Luo. The maximum number of cliques in graphs without long cycles. *Journal of Combinatorial Theory, Series B*, 128:219–226, 2018.

[68] L. Maherani and M. Shahsiah. Turán numbers of complete 3-uniform Berge-hypergraphs. *Graphs and Combinatorics*, 34(4):619–632, 2018.

[69] W. Mantel. Problem 28. *Wiskundige Opgaven*, 10(60-61):320, 1907.

[70] D. Mubayi. A hypergraph extension of Turán's theorem. *Journal of Combinatorial Theory, Series B*, 96(1):122–134, 2006.

[71] D. Mubayi and A. Suk. A survey of hypergraph Ramsey problems. *arXiv preprint arXiv:1707.04229*, 2017.

[72] D. Mubayi and J. Verstraëte. A survey of Turán problems for expansions. In *Recent Trends in Combinatorics*, pages 117–143. Springer, 2016.

[73] D. Mubayi and Y. Zhao. Forbidding complete hypergraphs as traces. *Graphs and Combinatorics*, 23(6):667–679, 2007.

[74] V. Nikiforov. The spectral radius of graphs without paths and cycles of specified length. *Linear Algebra and its Applications*, 432(9):2243–2256, 2010.

[75] B. Ning and X. Peng. Extensions of Erdős-Gallai Theorem and Luo's Theorem with Applications. *arXiv preprint arXiv:1801.09981*, 2018.

[76] C. Palmer, M. Tait, C. Timmons, and A. Z. Wagner. Turán numbers for Berge-hypergraphs and related extremal problems. *Discrete Mathematics*, 342(6):1553–1563, 2019.

[77] O. Pikhurko. Exact Computation of the Hypergraph Turán Function for Expanded Complete 2-Graphs. *Journal of Combinatorial Theory, Series B*, 103(2):220–225, 2013.

[78] F. Ramsey. On a Problem of Formal Logic. *Classic Papers in Combinatorics*, pages 1–24, 1987.

[79] N. Salia, C. Tompkins, Z. Wang, and O. Zamora. Ramsey numbers of berge-hypergraphs and related structures. *The Electronic Journal of Combinatorics*, 26(4):#P4.40, 2019.

[80] N. Sauer. On the density of families of sets. *Journal of Combinatorial Theory, Series A*, 13(1):145–147, 1972.

[81] S. Shelah. A combinatorial problem; stability and order for models and theories in infinitary languages. *Pacific Journal of Mathematics*, 41(1):247–261, 1972.

[82] A. Sidorenko. Asymptotic solution for a new class of forbidden r-graphs. *Combinatorica*, 9(2):207–215, 1989.

[83] P. Turán. On an external problem in graph theory. *Mat. Fiz. Lapok*, 48:436–452, 1941.

[84] V. N. Vapnik and A. Y. Chervonenkis. The uniform convergence of frequencies of the appearance of events to their probabilities. *Russian Academy of Sciences*, 181(4):781–783, 1968.

[85] N. C. Wormald. On the frequency of 3-connected subgraphs of planar graphs. *Bulletin of the Australian Mathematical Society*, 34(2):309–317, 1986.

[86] A. A. Zykov. On some properties of linear complexes. *Matematicheskii Sbornik*, 66(2):163–188, 1949.